

MA6001: Real Analysis -1

(Assignment 2: Metric and Normed Linear Spaces)

July - November, 2026

1. State TRUE or FALSE giving proper justification for each of the following statements.
 - (a) If X is a finite metric space, does it imply that $C(X)$, the space of continuous functions on X , is a finite dimensional normed linear space?
 - (b) If every countable closed set in a metric space (X, d) is complete, does it imply X is complete?
 - (c) The totally boundedness property is preserved by homeomorphism.
 - (d) Let $f_n : [0, 1] \rightarrow \mathbb{R}$ be defined by $f_n = \chi_{[0, 1/n]}$ and f_n converges point-wise to f . Then the set $\{f, f_n : n = 1, 2, \dots\}$ is compact in $B[0, 1]$.
 - (e) Let $f_n \in C^1[0, 1]$. Then it implies that the set $\{f_n : n = 1, 2, \dots\}$ is compact in $C[0, 1]$.
 - (f) Whether $\{x = (x_1, x_2, \dots) \in l^2 : |x_n| \leq \frac{1}{n}\}$ is totally bounded in l^2 ?
2. If every countable closed subset of a metric space X is complete, show that X is complete.
3. Show that a subset A of a metric space X is closed if and only if $A \cap K$ is compact for every compact set K in X .
4. Find a subset of l^∞ which is closed and bounded but not totally bounded.
5. Show that a subset A of a metric space (X, d) is totally bounded if and only if for every sequence (x_n) has a subsequence (x_{n_k}) satisfying $d(x_{n_k}, x_{n_{k+1}}) \leq 2^{-k}$.
6. Let K and F be two non-empty subsets of a metric space (X, d) . If K is compact and F closed, then show that $\text{dist}(K, F) > 0$, whenever $K \cap F = \emptyset$. Does the the same conclusion holds if K is closed but not compact?
7. A function $f : (X, d) \rightarrow \mathbb{R}$ is called lower semi-continuous if for each $\alpha \in \mathbb{R}$ the set $\{x \in X : f(x) > \alpha\}$ is open in X .
 - (a) Show that f is lower semi-continuous if and only if $f(x) \leq \liminf_{n \rightarrow \infty} f(x_n)$, whenever $x_n \rightarrow x$.
 - (b) If X is compact metric space, prove that every lower semi-continuous function is bounded below and attains its minimum.
8. Let $f : (X, d) \rightarrow \mathbb{R}$ be lower semi-continuous (LSC). Show that for every $x \in X$, and every sequence $x_n \rightarrow x$, implies $f(x) \leq \liminf_{n \rightarrow \infty} f(x_n)$.
9. If X is compact metric space, and let $f : X \rightarrow X$ satisfy $d(f(x), f(y)) = d(x, y)$ for $x, y \in X$. Show that f is an onto map. Is compactness of X is necessary?
10. If X is compact metric space, and let $f : X \rightarrow X$ satisfy $d(f(x), f(y)) \geq d(x, y)$ for $x, y \in X$. Show that f is an onto isometry.
11. Let X be a compact metric space, and let $f : X \rightarrow X$ be bijective and satisfy $d(f(x), f(y)) \leq d(x, y)$ for $x, y \in X$. Show that f is an isometry.
12. Let X be a compact metric space, and $\mathcal{F}$ is a subset of $(C(X))$.

- (a) Prove that an equicontinuous subset $\mathcal{F}$ is pointwise bounded if and only if $\mathcal{F}$ is uniformly bounded.
- (b) Prove that $\mathcal{F}$ is pointwise equicontinuous if and only if $\mathcal{F}$ uniformly equicontinuous.
13. Let X be a compact metric space, and (f_n) is a sequence in $(C(X))$.
- (a) Let (f_n) be equicontinuous and pointwise convergent. Show that f_n is uniformly convergent.
- (b) If (f_n) decreases pointwise to 0, show that (f_n) is equicontinuous.
- (c) If (f_n) is equicontinuous, show that $\{x \in X : (f_n(x)) \text{ converges}\}$ is a closed set in X .
14. For fixed $k > 0$ and $0 < \alpha \leq 1$, denote $\text{Lip}_k \alpha = \{f \in C[0, 1] : |f(x) - f(y)| \leq k|x - y|^\alpha\}$. Show that $\{f \in \text{Lip}_k \alpha : f(0) = 0\}$ is compact subset of $C[0, 1]$. Whether the set $\{f \in \text{Lip}_k \alpha : \int_0^1 f(t)dt = 1\}$ is compact?
15. Let $K(x, t)$ be a continuous function on the square $[0, 1] \times [0, 1]$. For $f \in C[0, 1]$, define $Tf(x) = \int_0^1 f(t)K(x, t)dt$. Show that T maps bounded sets into equicontinuous sets.
16. Let $f_n \in C[0, 1]$ be satisfying $\|f_n\|_\infty \leq 1$. Let $F_n(x) = \int_0^x f_n(t)dt$. Show that F_n has a convergent subsequence.
17. If $f \in B[0, 1]$, show that $B_n(f)(x) \rightarrow f(x)$ at each point of continuity of f .
18. Give an example of sequence of function $f_n \in C[0, 1]$, which decreases point wise to f but not uniformly.
19. For a given polynomial p and $\epsilon > 0$, show that there exists a polynomial q of rational coefficients such that $\|p - q\|_\infty < \epsilon$ on $[0, 1]$.
20. Let (x_i) be a sequence in $(0, 1)$ such that $\frac{1}{n} \sum_{i=1}^n x_i^k$ is convergent for each $k = 0, 1, 2, \dots$, then $\frac{1}{n} \sum_{i=1}^n f(x_i)$ is convergent for each $f \in C[0, 1]$.
21. For $f \in C[0, 1]$ and $\epsilon > 0$, show that there exists a polynomial p such that $\|f - p\|_\infty < \epsilon$ and $\|f' - p'\|_\infty < \epsilon$.
22. Let $f : [1, \infty) \rightarrow \mathbb{R}$ be continuous and $\lim_{x \rightarrow \infty} f(x)$ exists. For $\epsilon > 0$, show that there exists a polynomial p such that $|f(x) - p(1/x)| < \epsilon$ for all $x \geq 1$.