Assignment 1 (Measurable Sets-I)

1. State TRUE or FALSE giving proper justification for each of the following statements.
   (a) There exists a bounded set $A \subset \mathbb{R}$ such that $m^*(A) = 0$ but $m(\overline{A}) = 1$.
   (b) The set $E = \bigcup\limits_{x \in \mathbb{R}} (x + \mathbb{Q}) \setminus \mathbb{Q}$ is Lebesgue measurable.
   (c) There exists an unbounded subset A of $\mathbb{R}$ such that $m^*(A) = 5$.
   (d) There exists an open subset A of $\mathbb{R}$ such that $[\frac{1}{2}, \frac{3}{4}] \subset A$ and $m^*(A) = \frac{1}{4}$.
   (e) There exists an open subset A of $\mathbb{R}$ such that $m^*(A) < \frac{1}{5}$ but $A \cap (a, b) \neq \emptyset$ for all $a, b \in \mathbb{R}$ with $a < b$.
   (f) If A and B are open subsets of $\mathbb{R}$ such that $A \subsetneq B$, then it is necessary that $m^*(A) < m^*(B)$.

2. Examine whether $\mathcal{A}$ is a σ-algebra of subsets of $\mathbb{R}$, where
   (a) $\mathcal{A} = \{A \subset \mathbb{R} : m^*(A) = 0 \text{ or } m^*(\mathbb{R} \setminus A) = 0\}$.
   (b) $\mathcal{A} = \{A \subset \mathbb{R} : m^*(A) < +\infty \text{ or } m^*(\mathbb{R} \setminus A) < +\infty\}$.
   (c) $\mathcal{A} = \{A \subset \mathbb{R} : A \text{ or } \mathbb{R} \setminus A \text{ is an open subset of } \mathbb{R}\}$.

3. Let X be an uncountable set. Show that the class $\{\{x\} : x \in X\}$ generates the σ-algebra $\{A \subset X : A \text{ is countable or } X \setminus A \text{ is countable}\}$.

4. Let $\mathcal{S}$ be a class of subsets of a nonempty set X and let $A \subset X$. Show that $\sigma(\mathcal{S} \cap A) = \sigma(\mathcal{S}) \cap A$, where for each class $\mathcal{C}$ of subsets of X, $\mathcal{C} \cap A = \{C \cap A : C \in \mathcal{C}\}$.

5. Let X, Y be nonempty sets and let $f : X \to Y$. If $\mathcal{S}$ is a class of subsets of Y, then show that $\sigma(f^{-1}(\mathcal{S})) = f^{-1}(\sigma(\mathcal{S}))$, where for each class $\mathcal{C}$ of subsets of Y, $f^{-1}(\mathcal{C}) = \{f^{-1}(C) : C \in \mathcal{C}\}$.

6. If $\mathcal{S}$ is a class of subsets of a nonempty set X and if $A \in \sigma(\mathcal{S})$, then show that there exists a countable subclass $\mathcal{S}_0$ of $\mathcal{S}$ such that $A \in \sigma(\mathcal{S}_0)$.

7. Prove that every infinite σ-algebra on an infinite set is uncountable.

8. Let (X, S, μ) be a $\sigma-$ finite measure space. Show that for each $E \in S$ with $\mu(E) = \infty$, there exists a set $F \subset E$ such that $0 < \mu(F) < \infty$.

9. Show that $\mathcal{B}(\mathbb{R})$ is generated by each of the following classes.
   (a) $\{(a, +\infty) : a \in \mathbb{R}\}$
   (b) $\{(-\infty, a] : a \in \mathbb{Q}\}$
   (c) $\{[a, b) : a, b \in \mathbb{Q}, a < b\}$
   (d) $\{A \subset \mathbb{R} : A \text{ is compact}\}$

10. Let E be a Borel subset of $\mathbb{R}$ and let $x \in \mathbb{R}$. Show that $x + E$ is a Borel subset of $\mathbb{R}$.

11. Consider the outer measure μ^* on $\mathbb{R}$, where for each $A \subset \mathbb{R}$, $\mu^*(A) = \begin{cases} 0 & \text{if } A \text{ is countable,} \\ 1 & \text{if } A \text{ is uncountable.} \end{cases}$
    Determine all the μ^*-measurable subsets of $\mathbb{R}$.

12. If $\mathcal{S} = \{\emptyset, [1, 2]\}$ and if $\mu(\emptyset) = 0$, $\mu([1, 2]) = 1$, then determine the outer measure μ^* on $\mathbb{R}$ induced by the set function $\mu : \mathcal{S} \to [0, +\infty)$.
    Also, determine all the μ^*-measurable subsets of $\mathbb{R}$.

13. Let $(X, \mathcal{A})$ be a measurable space and let $\mu : \mathcal{A} \to [0, +\infty]$ be finitely additive with $\mu(\emptyset) = 0$. Show that μ is a measure on $\mathcal{A}$ if either of the following conditions is satisfied.
    (a) For every increasing sequence $\{A_n\}_{n=1}^{\infty}$ of sets in $\mathcal{A}$, $\lim\limits_{n \to \infty} \mu(A_n) = \mu(\bigcup\limits_{n=1}^{\infty} A_n)$.
    (b) For every decreasing sequence $\{A_n\}_{n=1}^{\infty}$ of sets in $\mathcal{A}$ with $\bigcap\limits_{n=1}^{\infty} A_n = \emptyset$, $\lim\limits_{n \to \infty} \mu(A_n) = 0$.

14. Let μ^* be an outer measure on a nonempty set X. Show that a subset E of X is μ^*-measurable iff for each $\varepsilon > 0$, there exists a μ^*-measurable set F such that $F \subset E$ and $\mu^*(E \setminus F) < \varepsilon$.

15. Let $f : [0, 2) \to \mathbb{R}$ be defined by $f(x) = \begin{cases} x^2 & \text{if } 0 \le x \le 1, \\ 3 - x & \text{if } 1 < x < 2. \end{cases}$
Find $m^*(A)$, where $A = f^{-1}((\frac{9}{16}, \frac{5}{4})) = \{x \in [0, 2) : f(x) \in (\frac{9}{16}, \frac{5}{4})\}$.

16. Let $B \subset A \subset \mathbb{R}$ such that $m^*(B) = 0$. Show that $m^*(A \setminus B) = m^*(A)$.

17. Let $A \subset \mathbb{R}$ such that $m^*(A) > 0$. Show that there exists $B \subset A$ such that B is bounded and $m^*(B) > 0$.

18. If G is a nonempty open subset of $\mathbb{R}$, then show that $m^*(G) > 0$.

19. Let A be a countable subset of $\mathbb{R}$ and let $B \subset \mathbb{R}$ such that $m^*(B) = 0$. Show that $m^*(A+B) = 0$.

20. Prove or disprove: A subset E of $\mathbb{R}$ is Lebesgue measurable iff $m^*(A \cup B) = m^*(A) + m^*(B)$ for each $A \subset E$ and for each $B \subset \mathbb{R} \setminus E$.

21. Let $A_n \subset \mathbb{R}$ for $n = 1, 2, ...$ such that $\sum_{n=1}^{\infty} m^*(A_n) < \infty$.
If $E = \{x \in \mathbb{R} : x \in A_n \text{ for infinitely many } n\}$, then show that $m(E) = 0$.

22. Show that a subset E of $\mathbb{R}$ is Lebesgue measurable iff $m^*(I) = m^*(I \cap E) + m^*(I \setminus E)$ for every bounded open interval I of $\mathbb{R}$.

23. Let $A \subset E \subset B \subset \mathbb{R}$ such that A, B are Lebesgue measurable and $m(A) = m(B) < \infty$. Show that E is Lebesgue measurable.
More generally, let $A \subset B \subset \mathbb{R}$ such that A is Lebesgue measurable and $m^*(B) = m(A) < \infty$. Show that B is Lebesgue measurable.

24. Let $A, B \subset \mathbb{R}$ such that $m^*(A) = 0$ and $A \cup B$ is Lebesgue measurable. Show that B is Lebesgue measurable.

25. Let $A, B \subset \mathbb{R}$ such that A is Lebesgue measurable and $m^*(A \triangle B) = 0$. Show that B is Lebesgue measurable.

26. Let $A \subset \mathbb{R}$ such that $A \cap B$ is Lebesgue measurable for every bounded subset B of $\mathbb{R}$. Show that A is Lebesgue measurable.

27. Let $E \subset \mathcal{M}(\mathbb{R})$ and let $A \subset \mathbb{R}$. Show that $m^*(E \cap A) + m^*(E \cup A) = m^*(E) + m^*(A)$.

28. Let I and J be disjoint open intervals in $\mathbb{R}$ and let $A \subset I$, $B \subset J$. Show that $m^*(A \cup B) = m^*(A) + m^*(B)$.

29. If $A \subset \mathbb{R}$ such that $m^*(A) = 0$, then show that $m^*(\{x^2 : x \in A\}) = 0$. Further, if $A \subset (0, 1)$, then show that $m^*(A^2) \le 2m^*(A)$.

30. Suppose $E \subset [0, 1]$ and $m^*(E) = 0$. Show that $E^2 = \{x^2 : x \in E\}$ is Lebesgue measurable.