MA642: Real Analysis -1

(Assignment 1: Metric and Normed Linear Spaces)
January - April, 2023

1. State TRUE or FALSE giving proper justification for each of the following statements.
   (a) There does not exist a monotone function $f : \mathbb{R} \to \mathbb{Q}$ which is onto.
   (b) There exists a monotone function $f : (0, \infty) \to \mathbb{R}$ such that each $c \in (0, \infty)$ satisfies $|f(c+) - f(c-)| = \frac{1}{c}$.
   (c) There exists a sequence of differential functions f_n on $(0, \infty)$ such that f_n' is uniformly convergent on $(0, \infty)$ but f_n is nowhere point-wise convergent.
   (d) There exists a metric space having exactly 36 open sets.
   (e) It is impossible to define a metric d on $\mathbb{R}$ such that only finitely many subsets of $\mathbb{R}$ are open in $(\mathbb{R}, d)$.
   (f) If A and B are open (closed) subsets of a normed vector space X, then $A + B = \{a + b : a \in A, b \in B\}$ is open (closed) in X.
   (g) If A and B are closed subsets of $[0, \infty)$ (with the usual metric), then $A + B$ is closed in $[0, \infty)$.
   (h) It is possible to define a metric d on $\mathbb{R}$ such that the sequence $(1, 0, 1, 0, ...)$ converges in $(\mathbb{R}, d)$.
   (i) It is possible to define a metric d on $\mathbb{R}^2$ such that $((\frac{1}{n}, \frac{n}{n+1}))$ is not a Cauchy sequence in $(\mathbb{R}^2, d)$.
   (j) It is possible to define a metric d on $\mathbb{R}^2$ such that in $(\mathbb{R}^2, d)$, the sequence $((\frac{1}{n}, 0))$ converges but the sequence $((\frac{1}{n}, \frac{1}{n}))$ does not converge.
   (k) Let $A \subset (1, \infty)$ be a closed set. Then $A^2 := \{a^2 : a \in A\}$ is a closed set.
   (l) Let $A_n = \{(x, y) \in \mathbb{R}^2 : 0 < \frac{1}{x} < y < \frac{1}{n}\}$. Whether the set $\bigcap_{n=1}^{\infty} A_n$ is open/closed?
   (m) There exist a set $A \subset (\mathbb{R}, u)$ such that $\delta\left(A^o \cup \{0\}\right) = 0$ but $\delta\left((\bar{A})^o\right) = 1$, where δ stands for diameter.
   (n) If (x_n) is a sequence in a complete normed vector space X such that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$, then (x_n) must converge in X.
   (o) If (f_n) is a sequence in $C[0, 1]$ such that $|f_{n+1}(x) - f_n(x)| \leq \frac{1}{n^2}$ for all $n \in \mathbb{N}$ and for all $x \in [0, 1]$, then there must exist $f \in C[0, 1]$ such that $\int_0^1 |f_n(x) - f(x)|\, dx \to 0$ as $n \to \infty$.
   (p) If (x_n) is a Cauchy sequence in a normed vector space, then $\lim_{n\to\infty} \|x_n\|$ must exist.
   (q) $\{f \in C[0, 1] : \|f\|_1 \leq 1\}$ is a bounded subset of the normed vector space $(C[0, 1], \|\cdot\|_\infty)$.
   (r) The space $(C^1[0, 1], \|\ .\ \|)$, where $\|f\| = (\|f\|_2^2 + \|f'\|_2^2)^{\frac{1}{2}}$ is complete.
   (s) Let $f \in C^1[0, 1]$ and $\|f\| = \|f'\|_2 + \|f\|_\infty$. Then $(C^1[0, 1], \|\ .\ \|)$ is complete.
   (t) Let $f \in C^1[0, 1]$. Then $\|f\| = \min(\|f'\|_2,\ \|f\|_\infty)$ defines a norm on $C^1[0, 1]$.
   (u) Let $X = \{f \in C^1[0, 1] : f(0) = 0\}$. Then $\|f\| = \|f'\|_2$ is a norm on $C^1[0, 1]$ but not complete
   (v) For $x, y \in l^\infty$, $d(x, y) = \min\{1, \limsup |x_n - y_n|\}$ define a metric on l^∞.
   (w) The sequence $f_n(t) = e^{-n^2 \sin \pi t}$ converge uniformly to 0 on $(0, 1)$.
2. What is the cardinality of the set $\{f : \mathbb{R} \to \mathbb{R},\ f \text{ is nowhere continuous}\}$?
3. For a monotone increasing function $f : [a, b] \to \mathbb{R}$, define $g(x) = \sup\{f(y) : y < x\}$. If f has limit at c, then show that $f(c) = g(c)$.
4. Examine whether (X, d) is a metric space, where

(a) $X=\mathbb{R}$ and $d(x,y)=\frac{|x-y|}{1+|xy|}$ for all $x,y\in\mathbb{R}$.
(b) $X=\mathbb{R}$ and $d(x,y)=|x-y|^p$ for all $x,y\in\mathbb{R}$ $(0<p<1)$.
(c) $X=\mathbb{R}$ and $d(x,y)=\min\{\sqrt{|x-y|},|x-y|^2\}$ for all $x,y\in\mathbb{R}$.
(d) $X=\mathbb{R}$ and for all $x,y\in\mathbb{R}$, $d(x,y)=\begin{cases}1+|x-y| & \text{if exactly one of } x \text{ and } y \text{ is positive,}\\ |x-y| & \text{otherwise.}\end{cases}$
(e) $X=\mathbb{R}^2$ and $d(x,y)=(|x_1-y_1|+|x_2-y_2|^{\frac{1}{2}})^{\frac{1}{2}}$ for all $x=(x_1,x_2),y=(y_1,y_2)\in\mathbb{R}^2$.
(f) $X=\mathbb{R}^n$ and $d(x,y)=[(x_1-y_1)^2+\frac{1}{2}(x_2-y_2)^2+\cdots+\frac{1}{n}(x_n-y_n)^2]^{\frac{1}{2}}$ for all $x=(x_1,...,x_n)$, $y=(y_1,...,y_n)\in\mathbb{R}^n$.
(g) $X=\mathbb{C}$ and for all $z,w\in\mathbb{C}$, $d(z,w)=\begin{cases}\min\{|z|+|w|,|z-1|+|w-1| & \text{if } z\neq w,\\ 0 & \text{if } z=w.\end{cases}$
(h) $X=\mathbb{C}$ and for all $z,w\in\mathbb{C}$, $d(z,w)=\begin{cases}|z-w| & \text{if } \frac{z}{|z|}=\frac{w}{|w|},\\ |z|+|w| & \text{otherwise.}\end{cases}$
(i) $X=\mathbb{C}$ and $d(z,w)=\frac{2|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}}$ for all $z,w\in\mathbb{C}$.
(j) $X=$ The class of all finite subsets of a nonempty set and $d(A,B)=$ The number of elements of the set $A\triangle B$ (the symmetric difference of A and B).

5. Let $1\leq p\leq\infty$ and $d_i\,;i=1,2$ be two metric on a non-emply set X. Show that $d_p=(d_1^p+d_2^p)^{1/p}$ is a metric on X for $1\leq p<\infty$. Whether $d_\infty=\max\{d_1,d_2\}$ is a metric on X?
6. Examine whether $\|\cdot\|$ is a norm on $\mathbb{R}^2$, where for each $(x,y)\in\mathbb{R}^2$,
   (a) $\|(x,y)\|=(|x|^p+|y|^p)^{\frac{1}{p}}$, where $0<p<1$.
   (b) $\|(x,y)\|=\sqrt{\frac{x^2}{9}+\frac{y^2}{4}}$.
   (c) $\|(x,y)\|=\begin{cases}\sqrt{x^2+y^2} & \text{if } xy\geq 0,\\ \max\{|x|,|y|\} & \text{if } xy<0.\end{cases}$

7. Let $\|f\|=\min\{\|f\|_\infty,2\|f\|_1\}$ for all $f\in C[0,1]$. Prove that $\|\cdot\|$ is not a norm on $C[0,1]$.
8. Let $D=\{z\in\mathbb{C}:|z|<1\}$. Let X be the class of all functions f which are analytic on D and continuous on $\bar{D}$. Define $\|f\|=\sup\{|f(e^{it})|:0\leq t\leq 2\pi\}$. Show that $(X,\|\ .\ \|)$ is complete.
9. Let X be a normed linear space. Prove that norm of any $x\in X$, can be expressed as $\|x\|=\inf\{|\alpha|:\ \alpha\in\mathbb{C}\setminus\{0\}\text{ with }\|x\|\leq|\alpha|\}$.
10. If $\mathbf{x}\in\mathbb{R}^n$, then show that $\lim\limits_{p\to\infty}\|\mathbf{x}\|_p=\|\mathbf{x}\|_\infty$.
11. Let $(X,\|\cdot\|)$ be a normed linear space. Show that $\|x\|=\sup\{|\alpha|:\ |\alpha|<\|x\|\}$.
12. Let d be a metric on a real vector space X satisfying the following two conditions:
    (i) $d(x+z,y+z)=d(x,y)$ for all $x,y,z\in X$,
    (ii) $d(\alpha x,\alpha y)=|\alpha|d(x,y)$ for all $x,y\in X$ and for all $\alpha\in\mathbb{R}$.
    Show that there exists a norm $\|\cdot\|$ on X such that $d(x,y)=\|x-y\|$ for all $x,y\in X$.
13. Let f be a non-negative function on a linear space X such that $f(\alpha x)=|\alpha|f(x)$ for all $\alpha\in\mathbb{C}$. Show that f is norm on X if and only if f is a convex map which can vanish at most at one point.
14. Let $f:(X,d)\to[0,1]$ be continuous map. Show that $f^{-1}(0)$ is a closed G_δ set.
15. If $1\leq p<q\leq\infty$, then show that $\|x\|_q\leq\|x\|_p$ for all $x\in\ell^p$.
16. For $x=(x_n)\in l^2$, write $\|x\|=(\sum\limits_{n=1}^{\infty}a_n|x_n|^2)^{1/2}$. Find all possible sequences (a_n) such that $\|\cdot\|$ is a norm on l^2.
17. Let $\mathbb{R}^\infty$ be the real vector space of all sequences in $\mathbb{R}$, where addition and scalar multiplication are defined componentwise. Let $d((x_n),(y_n))=\sum\limits_{n=1}^{\infty}\frac{1}{2^n}\cdot\frac{|x_n-y_n|}{1+|x_n-y_n|}$ for all $(x_n),(y_n)\in\mathbb{R}^\infty$. Show that d is a metric on $\mathbb{R}^\infty$ but that no norm on $\mathbb{R}^\infty$ induces d.

18. Let $(X, \|\cdot\|)$ be a nonzero normed vector space. Consider the metrics d_1, d_2 and d_3 on X:

$$d_1(x, y) := \min\{1, \|x - y\|\},$$
$$d_2(x, y) := \frac{\|x - y\|}{1 + \|x - y\|},$$
$$d_3(x, y) := \begin{cases} 1 + \|x - y\| & \text{if } x \neq y, \\ 0 & \text{if } x = y, \end{cases}$$

for all $x, y \in X$. Prove that none of d_1, d_2 and d_3 is induced by any norm on X.

19. Let X be a normed vector space containing more than one point, let $x, y \in X$ and let $\varepsilon, \delta > 0$. If $B_\varepsilon[x] = B_\delta[y]$, show that $x = y$ and $\varepsilon = \delta$. Does the result remain true if X is assumed to be a metric space? Justify.
20. Let $A = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 < 1\}$ and $B = \{(x, y, z) \in \mathbb{R}^3 : z = 0\}$. Examine whether $A \cap B$ is a closed/an open subset of $\mathbb{R}^3$ with respect to the usual metric on $\mathbb{R}^3$.
21. Let F_n be a sequence of closed sets in $\mathbb{R}$ such that $F_n \subset (n, n+1]$ and $F_n \cap F_m = \emptyset$, whenever $m \neq n$. Show that $F = \bigcup\limits_{n=1}^{\infty} F_n$ is a closed set in $\mathbb{R}$.
22. For all $x, y \in \mathbb{R}$, let $d_1(x, y) = |x - y|$, $d_2(x, y) = \min\{1, |x - y|\}$ and $d_3(x, y) = \frac{|x-y|}{1+|x-y|}$. If G is an open set in any one of the three metric spaces $(\mathbb{R}, d_i)$ $(i = 1, 2, 3)$, then show that G is also open in the other two metric spaces.
23. Let X be a normed vector space and let $Y (\neq X)$ be a subspace of X. Show that Y is not open in X.
24. Let (x_n) and (y_n) be Cauchy sequences in a metric space (X, d). Show that the sequence $(d(x_n, y_n))$ is convergent.
25. Let d_o be the discrete metric on non-empty set X. Show that (X, d_o) is complete.
26. Let (x_n) be a sequence in a complete metric space (X, d) such that $\sum\limits_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty$. Show that (x_n) converges in (X, d).

27. Let (x_n) be a sequence in a metric space X such that each of the subsequences (x_{2n}), (x_{2n-1}) and (x_{3n}) converges in X. Show that (x_n) converges in X.

28. Show that the following are incomplete metric spaces.
    (a) $(\mathbb{N}, d)$, where $d(m, n) = |\frac{1}{m} - \frac{1}{n}|$ for all $m, n \in \mathbb{N}$
    (b) $((0, \infty), d)$, where $d(x, y) = |\frac{1}{x} - \frac{1}{y}|$ for all $x, y \in (0, \infty)$
    (c) $(\mathbb{R}, d)$, where $d(x, y) = |\frac{x}{1+|x|} - \frac{y}{1+|y|}|$ for all $x, y \in \mathbb{R}$
    (d) $(\mathbb{R}, d)$, where $d(x, y) = |e^x - e^y|$ for all $x, y \in \mathbb{R}$

29. Examine whether the following metric spaces are complete.
    (a) $([0, 1), d)$, where $d(x, y) = |\frac{x}{1-x} - \frac{y}{1-y}|$ for all $x, y \in [0, 1)$
    (b) $((-1, 1), d)$, where $d(x, y) = |\tan\frac{\pi x}{2} - \tan\frac{\pi y}{2}|$ for all $x, y \in (-1, 1)$
    (c) $((0, 2], d)$, where $d(x, y) = |\frac{1}{x} - \frac{1}{y}|$

30. For $X (\neq \emptyset) \subset \mathbb{R}$, let $d(x, y) = \frac{|x-y|}{1+|x-y|}$ for all $x, y \in X$. Examine the completeness of the metric space (X, d), where X is
    (a) $[0, 1] \cap \mathbb{Q}$.
    (b) $[-1, 0] \cup [1, \infty)$.
    (c) $\{n^2 : n \in \mathbb{N}\}$.

31. Examine whether the sequence (f_n) is convergent in $(C[0,1], d_\infty)$, where for all $n \in \mathbb{N}$ and for all $t \in [0,1]$,

(a) $f_n(t) = \frac{nt^2}{1+nt}$.

(b) $f_n(t) = 1 + t + \frac{t^2}{2!} + \cdots + \frac{t^n}{n!}$.

(c) $f_n(t) = \begin{cases} nt & \text{if } 0 \le t \le \frac{1}{n}, \\ \frac{1}{nt} & \text{if } \frac{1}{n} < t \le 1. \end{cases}$

(d) $f_n(t) = \begin{cases} nt & \text{if } 0 \le t \le \frac{1}{n}, \\ \frac{n}{n-1}(1-t) & \text{if } \frac{1}{n} < t \le 1. \end{cases}$

32. Let K and F be two non-empty subsets of a metric space (X,d). If K is compact and F closed, then show that $\text{dist}(K,F) > 0$, whenever $K \cap F = \emptyset$.
33. Find the point-wise limit of the sequence $f_n(t) = e^{-nt^2} \sin nt$. Examine for uniform convergence of f_n on $\mathbb{R}$.
34. Let $f_n, f : \mathbb{R} \to (0,\infty)$ be such that $f_n \to f$ uniformly on $\mathbb{R}$. Examine for $e^{f_n} \to e^f$ uniformly on $\mathbb{R}$.
35. Let $f_n(t) = \sqrt{t^2+n}$. Examine for the uniform convergence of f'_n on $\mathbb{R}$.
36. Let X be the class of all continuous functions $f : \mathbb{R} \to \mathbb{C}$ such that for each $\epsilon > 0$, there exists a compact set $K \subset \mathbb{R}$ such that $|f(x)| < \epsilon$, for all $x \in \mathbb{R} \setminus K$. Show that $(X, \|\ .\ \|_\infty)$ is complete.
37. Let (x_n) be a sequence in a normed linear space X which converges to a non-zero vector $x \in X$. Show that $\frac{x_1+\cdots+x_n}{n^\alpha} \to x$ if and only if $\alpha = 1$. What are admissible values of α if $x_n \to 0$?
38. Let $X = C[0,1]$ be the space all the continuous functions on interval $[0,1]$. Prove that norms $\|\ .\ \|_\infty$ and $\|\ .\ \|_1$ on X are not equivalent.
39. Let $C^1[0,1]$ denote the space of all continuously differentiable functions on $[0,1]$. For $f \in C^1[0,1]$, define $\|f\| = \|f\|_\infty + \|f'\|_\infty$. Show that space $(C^1[0,1], \|\ .\ \|)$ is a Banach space.
40. Let $1 \le p < \infty$. Let X_p be a class of all the Riemann integrable functions on $[0,1]$. Prove that $\|f\|_p = \left(\int_0^1 |f|^p\right)^{\frac{1}{p}} < \infty$. Prove that $(X_p, \ \|\ .\ \|_p)$ is a normed linear space but not complete.
41. Let M be a subspace of a normed linear space X. Then show that M is closed if and only if $\{y \in M : \ \|y\| \le 1\}$ is closed in X.
42. Let $T : (C[0,\frac{\pi}{2}], \|.\|_\infty) \to (C[0,\frac{\pi}{2}], \|.\|_\infty)$ be defined by $(Tf)(x) = \int_{s=0}^{x} f(s) \sin s ds$. Show that T is not a contraction but T^2 is a contraction.
43. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuous and let there exist $\alpha > 0$ such that $\|f(\mathbf{x}) - f(\mathbf{y})\| \ge \alpha\|\mathbf{x} - \mathbf{y}\|$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Show that $f(\mathbb{R}^n)$ is complete.
44. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a contraction and let $g(\mathbf{x}) = \mathbf{x} - f(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$. Show that $g : \mathbb{R}^n \to \mathbb{R}^n$ is one-one and onto. Also, show that both g and $g^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ are continuous.