Assignment 1

1. Let $X = C[0, 1]$ be the space all the continuous functions on interval $[0, 1]$. Prove that norms $\| \cdot \|_\infty$ and $\| \cdot \|_1$ on $X$ are not equivalent.

2. Let $C^1[0, 1]$ denote the space of all continuously differentiable functions on $[0, 1]$.
   For $f \in C^1[0, 1]$, define $\| f \| = \| f \|_\infty + \| f' \|_\infty$. Show that space $(C^1[0, 1], \| \cdot \|)$ is a Banach space.

3. Does space $(C^1[0, 1], \| \cdot \|)$, where $\| f \| = \left( \int_0^1 |f|^2 + \int_0^1 |f'|^2 \right)^{\frac{1}{2}}$ a Banach space?

4. Let $f \in C^1[0, 1]$. Write $\| f \| = \| f' \|_2 + \| f \|_\infty$. Whether $(C^1[0, 1], \| \cdot \|)$ is a Banach space?

5. Let $f \in C^1[0, 1]$. Does $\| f \| = \min (\| f' \|_2, \| f \|_\infty)$ defines a norm on $C^1[0, 1]$?

6. Let $X = \{ f \in C^1[0, 1] : f(0) = 0 \}$. Then $\| f \| = \| f' \|$ is a norm on $C^1[0, 1]$. Whether $(X, \| \cdot \|)$ is a Banach space?

7. Let $(V, \| \cdot \|)$ be a normed linear space and $X$ is the space of all the continuous functions from $[0, 1]$ to $V$ with $\| f \|_\infty = \sup_{t \in [0, 1]} \| f(t) \|$. Prove that $(X, \| \cdot \|_\infty)$ is a Banach space.

8. Let $X$ be the class of all continuous functions $f : \mathbb{R} \to \mathbb{C}$ such that for each $\epsilon > 0$, there exists a compact set $K \subset \mathbb{R}$ such that $|f(x)| < \epsilon$, for all $x \in \mathbb{R} \setminus K$. Show that $(X, \| \cdot \|_\infty)$ is a Banach space.

9. Let $1 \leq p < \infty$. Let $X_p$ be a class of all the Riemann integrable functions on $[0, 1]$. Prove that $\| f \|_p = \left( \int_0^1 |f|^p \right)^{\frac{1}{p}} < \infty$. Prove that $(X_p, \| \cdot \|_p)$ is a normed linear space but not complete.

10. Let $1 \leq p < \infty$. Let $L^p[0, 1] = \{ f : [0, 1] \to \mathbb{C}, f$ is Lebesgue measurable $\}$ with $\| f \|_p = \left( \int_0^1 |f|^p \right)^{\frac{1}{p}} < \infty$. show that $L^p[0, 1]$ is proper dense subspace of $L^1[0, 1]$, whenever $1 < p < \infty$.

11. Let $1 \leq p < \infty$. Let $L^p(\mathbb{C}) = \{ f : \mathbb{C} \to \mathbb{C}, f$ is Lebesgue measurable $\}$ with $\| f \|_p = \left( \int_{\mathbb{C}} |f|^p \right)^{\frac{1}{p}} < \infty$. Let $C_c(\mathbb{C})$ be the class of all compactly supported functions on $\mathbb{C}$. Prove that $C_c(\mathbb{C})$ is proper dense subspace of $L^p(\mathbb{C})$, whenever $1 \leq p < \infty$. Whether $C_c(\mathbb{C})$ is a dense subspace of $L^\infty(\mathbb{C})$?
12. Let \((x_n)\) be a sequence in a normed linear space \(X\) which converges to a non-zero vector \(x \in X\). Show that
\[
\frac{x_1 + \cdots + x_n}{n^\alpha} \to x
\]
if and only if \(\alpha = 1\). If the sequence \(x_n \to 0\), prove that
\[
\frac{x_1 + \cdots + x_n}{n^\alpha} \to 0, \text{ for all } \alpha \geq 1.
\]

13. Prove that \(l^\infty(\mathbb{N}) = \{x = (x_1, x_2, \ldots) : \|x\|_\infty = \sup_j |x_j|\}\) is a Banach space but not separable.

14. Let \(M\) be a subspace of a normed linear space \(X\). Then show that \(M\) is closed if and only if \(\{y \in M : \|y\| \leq 1\}\) is closed in \(X\).

15. Let \(D = \{z \in \mathbb{C} : |z| < 1\}\). Let \(X\) be the class of all functions \(f\) which are analytic on \(D\) and continuous on \(\bar{D}\). Define \(\|f\|_\infty = \sup\{|f(e^{it})| : 0 \leq t \leq 2\pi\}\). Prove that \((X, \|\cdot\|_\infty)\) is a Banach space.

16. A normed linear space \(X\) is finite dimensional if and only if any ball \(B_r(x)\) in \(X\) is compact.

17. Show that unit ball \(\{x \in l^1(\mathbb{N}) : \|x\|_1 \leq 1\}\) is not compact in \(l^1(\mathbb{N})\), without using the statement of Q16.

18. Show that unit ball \(\{x \in l^\infty(\mathbb{N}) : \|x\|_\infty \leq 1\}\) is not compact \(l^\infty(\mathbb{N})\), without using the statement of Q16.

19. Let \(M\) be a closed subspace of a normed linear space \(X\). Prove that projection \(\pi : X \to X/M\) defined by \(\pi(x) = \tilde{x}\) is a continuous map.

20. Let \(X\) be a normed linear space. Prove that norm of any \(x \in X\), can be expressed as \(\|x\| = \inf \{\|\alpha\| : \alpha \in \mathbb{C} \setminus \{0\} \text{ with } \|x\| \leq |\alpha|\}\).

21. Let \(M\) be a closed subspace of a normed linear space \(X\). Then show that \(X\) is separable if and only if \(M\) and \(X/M\) both are separable.

22. Does there exist a separable Banach space which has no Schauder basis?

23. Let \(X\) be a separable Banach space. Prove that there exists a closed subspace \(M\) of \(l^1(\mathbb{N})\) such that \(X\) is isomorphic to \(l^1(\mathbb{N})/M\).