ME 101: Engineering Mechanics

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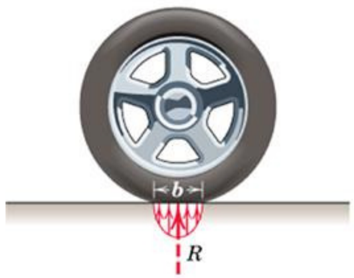
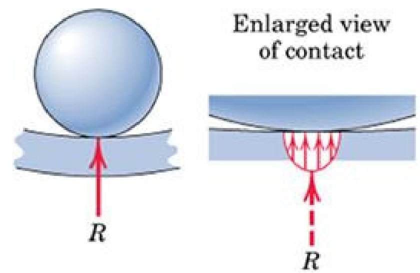
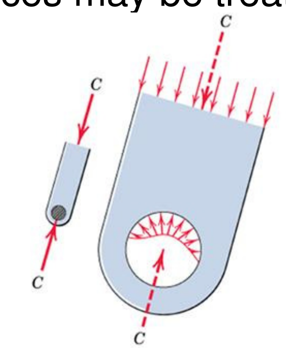
Center of Mass and Centroids

Concentrated Forces: If dimension of the contact area is negligible compared to other dimensions of the body → the contact forces may be treated as Concentrated Forces

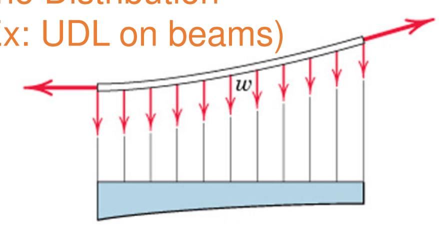


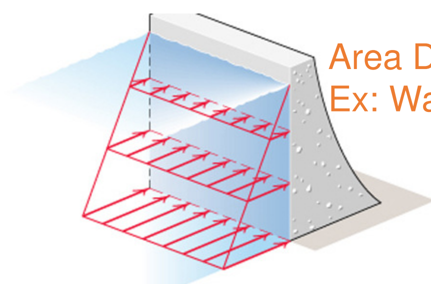




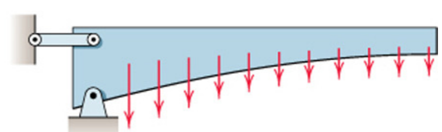
Distributed Forces: If forces are applied over a region whose dimension is not negligible compared with other pertinent dimensions → proper distribution of contact forces must be accounted for to know intensity of force at any location.

Line Distribution
(Ex: UDL on beams)





Area Distribution
Ex: Water Pressure



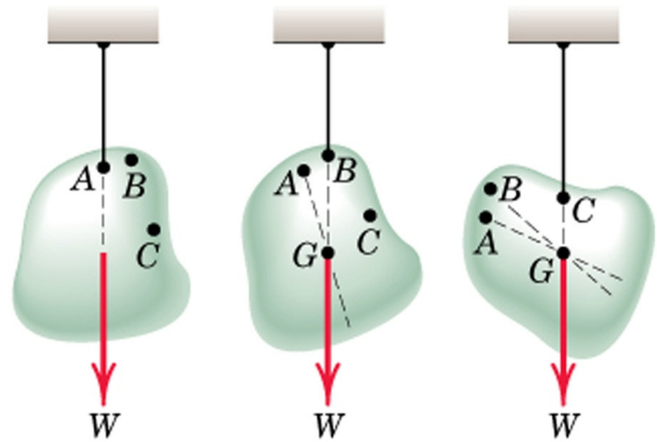
Body Distribution
(Ex: Self weight)

Center of Mass and Centroids

Center of Mass

A body of mass *m* in equilibrium under the action of tension in the cord, and resultant *W* of the gravitational forces acting on all particles of the body.

- The resultant is collinear with the cord



Suspend the body from different points on the body

- Dotted lines show lines of action of the resultant force in each case.
- These lines of action will be concurrent at a single point G

As long as dimensions of the body are smaller compared with those of the earth.
- we assume uniform and parallel force field due to the gravitational attraction of the earth.

The unique **Point G** is called the Center of Gravity of the body (CG)

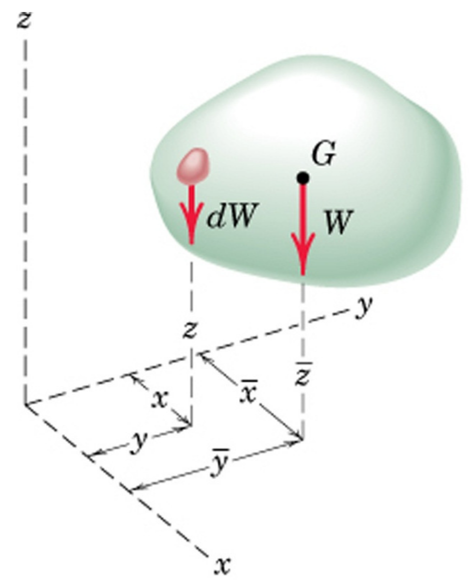
Center of Mass and Centroids

Determination of CG

- Apply Principle of Moments

Moment of resultant gravitational force W about any axis equals sum of the moments about the same axis of the gravitational forces dW acting on all particles treated as infinitesimal elements.

Weight of the body $W = \int dW$
Moment of weight of an element (dW) @ x-axis = ydW
Sum of moments for all elements of body = $\int ydW$
From Principle of Moments: $\int ydW = \bar{y}\ W$



$$\bar{x} = \frac{\int x dW}{W} \quad \bar{y} = \frac{\int y dW}{W} \quad \bar{z} = \frac{\int z dW}{W}$$

→ Numerator of these expressions represents the sum of the moments;
Product of W and corresponding coordinate of G represents the moment of the sum → Moment Principle.

$$\bar{x} = \frac{\int x dW}{W} \qquad \bar{y} = \frac{\int y dW}{W} \qquad \bar{z} = \frac{\int z dW}{W}$$

Center of Mass and Centroids

Determination of CG

Substituting $W = mg$ and $dW = gdm$

→ $$\bar{x} = \frac{\int x dm}{m} \qquad \bar{y} = \frac{\int y dm}{m} \qquad \bar{z} = \frac{\int z dm}{m}$$

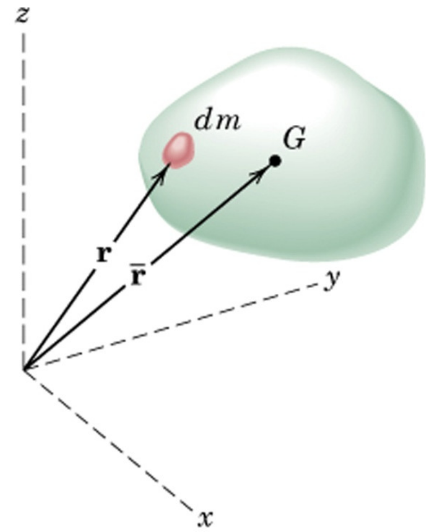
In vector notations:

Position vector for elemental mass: $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

Position vector for mass center G: $\bar{\mathbf{r}} = \bar{x}\mathbf{i} + \bar{y}\mathbf{j} + \bar{z}\mathbf{k}$

→ $$\bar{\mathbf{r}} = \frac{\int \mathbf{r} dm}{m}$$

The above equations are the components of this single vector equation



Density $\boldsymbol{\rho}$ of a body = **mass per unit volume**

→ Mass of a differential element of volume dV → $dm = \rho dV$

→ ρ may not be constant throughout the body

$$\bar{x} = \frac{\int x\rho dV}{\int \rho dV} \qquad \bar{y} = \frac{\int y\rho dV}{\int \rho dV} \qquad \bar{z} = \frac{\int z\rho dV}{\int \rho dV}$$

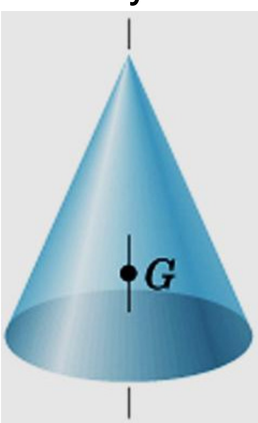
Center of Mass and Centroids

Center of Mass: Following equations independent of *g*

$$\bar{x} = \frac{\int x\,dm}{m} \qquad \bar{y} = \frac{\int y\,dm}{m} \qquad \bar{z} = \frac{\int z\,dm}{m} \qquad \bar{\mathbf{r}} = \frac{\int \mathbf{r}\,dm}{m} \qquad \bar{x} = \frac{\int x\rho\,dV}{\int \rho\,dV} \qquad \bar{y} = \frac{\int y\rho\,dV}{\int \rho\,dV} \qquad \bar{z} = \frac{\int z\rho\,dV}{\int \rho\,dV}$$

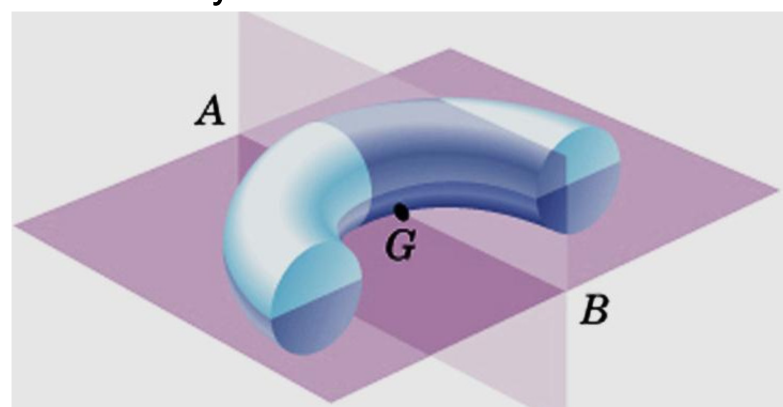
→ They define a unique point, which is a function of distribution of mass
→ This point is Center of Mass (CM)
→ CM coincides with CG as long as gravity field is treated as uniform and parallel
→ CG or CM may lie outside the body

CM always lie on a line or a plane of symmetry in a homogeneous body



Right Circular Cone
CM on central axis

Half Right Circular Cone
CM on vertical plane of symmetry



Half Ring
CM on intersection of two planes of symmetry (line AB)

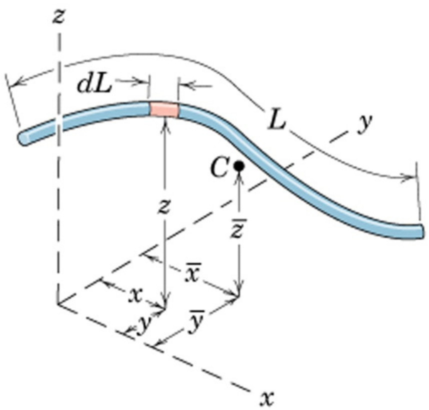
$$\bar{x} = \frac{\int x dm}{m} \quad \bar{y} = \frac{\int y dm}{m} \quad \bar{z} = \frac{\int z dm}{m}$$

Center of Mass and Centroids

Centroids of Lines, Areas, and Volumes

Centroid is a geometrical property of a body

→ When density of a body is uniform throughout, centroid and CM coincide



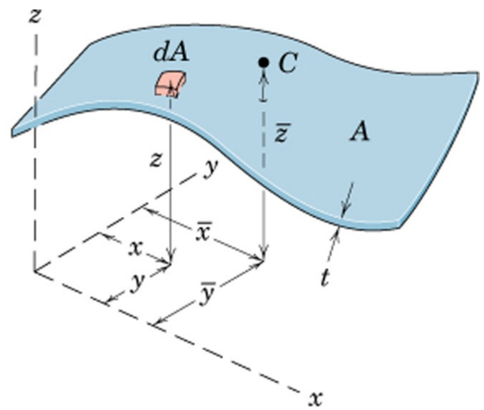
Lines: Slender rod, Wire

Cross-sectional area = A

ρ and A are constant over L

$dm = \rho A dL$; Centroid = CM

$$\bar{x} = \frac{\int x dL}{L} \quad \bar{y} = \frac{\int y dL}{L} \quad \bar{z} = \frac{\int z dL}{L}$$



Areas: Body with small but constant thickness t

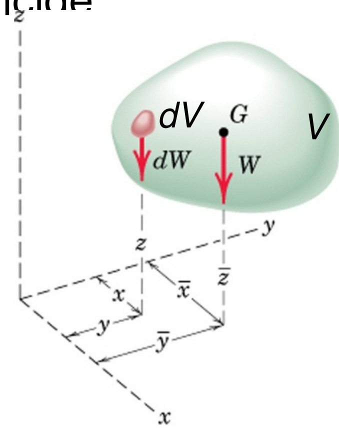
Cross-sectional area = A

ρ and A are constant over A

$dm = \rho t dA$; Centroid = CM

$$\bar{x} = \frac{\int x dA}{A} \quad \bar{y} = \frac{\int y dA}{A} \quad \bar{z} = \frac{\int z dA}{A}$$

Numerator = First moments of Area



Volumes: Body with volume V

ρ constant over V

$dm = \rho dV$ Centroid = CM

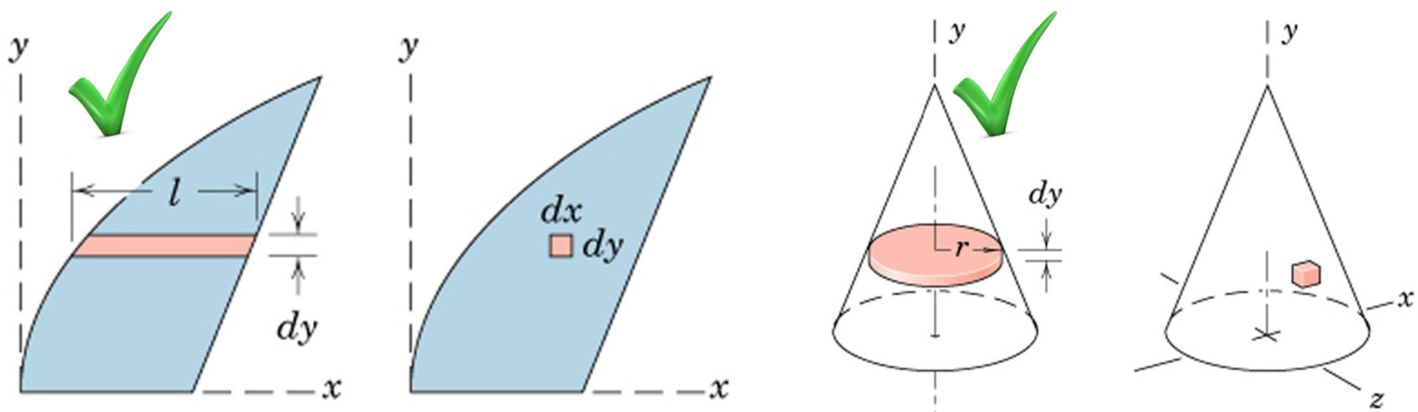
$$\bar{x} = \frac{\int x dV}{V} \quad \bar{y} = \frac{\int y dV}{V} \quad \bar{z} = \frac{\int z dV}{V}$$

Center of Mass and Centroids

Centroids of Lines, Areas, and Volumes

Guidelines for Choice of Elements for Integration

- Order of Element Selected for Integration
  A first order differential element should be selected in preference to a higher order element → only one integration should cover the entire figure



$A = \int dA = \int l\,dy$ $\quad$ $A = \int\int dx\,dy$ $\quad$ $V = \int dV = \int \pi r^2\,dy$ $\quad$ $V = \int\int\int dxdydz$

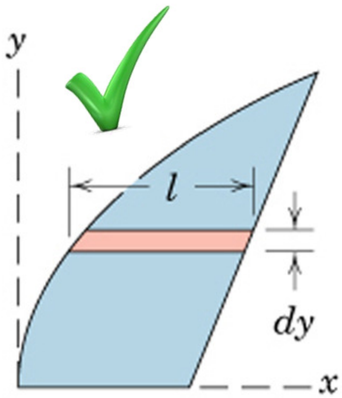
Center of Mass and Centroids

Centroids of Lines, Areas, and Volumes

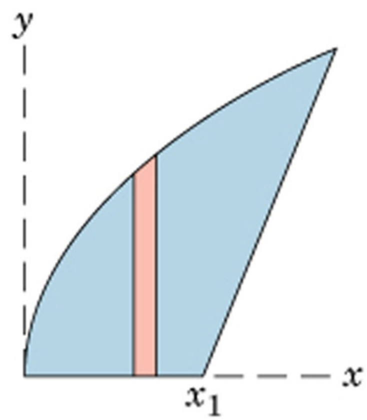
Guidelines for Choice of Elements for Integration

- Continuity

  Choose an element that can be integrated in one continuous operation to cover the entire figure → the function representing the body should be continuous
  → only one integral will cover the entire figure



Continuity in the expression for the width of the strip



Discontinuity in the expression for the height of the strip at $x = x_1$

Center of Mass and Centroids

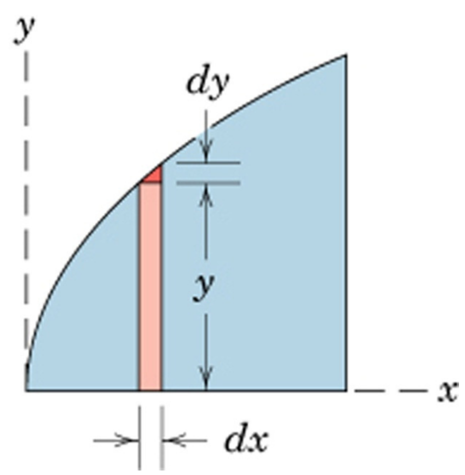
Centroids of Lines, Areas, and Volumes

Guidelines for Choice of Elements for Integration

- Discarding Higher Order Terms

  Higher order terms may always be dropped compared with lower order terms

Vertical strip of area under the curve is given by the first order term → $dA = ydx$
The second order triangular area $0.5dxdy$ may be discarded



Center of Mass and Centroids

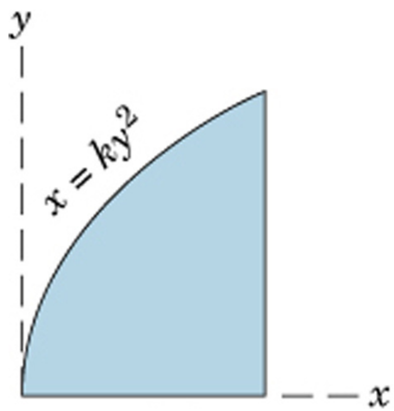
Centroids of Lines, Areas, and Volumes

Guidelines for Choice of Elements for Integration

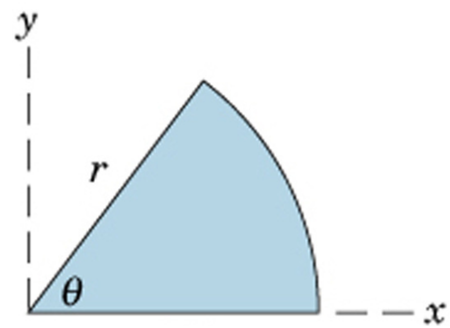
- Choice of Coordinates

  Coordinate system should best match the boundaries of the figure

  → easiest coordinate system that satisfies boundary conditions should be chosen



Boundaries of this area (not circular) can be easily described in rectangular coordinates



Boundaries of this circular sector are best suited to polar coordinates

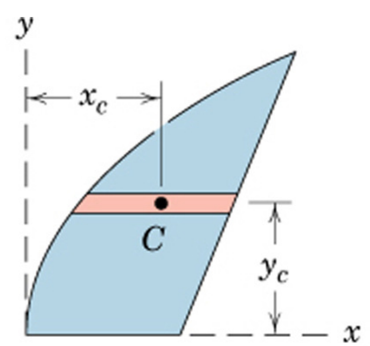
Center of Mass and Centroids

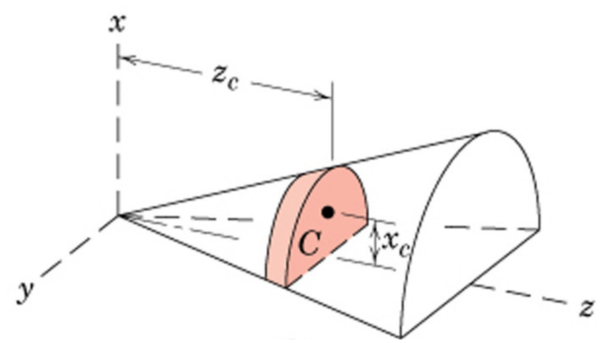
Centroids of Lines, Areas, and Volumes

Guidelines for Choice of Elements for Integration

- Centroidal Coordinate of Differential Elements

  While expressing moment of differential elements, take coordinates of the centroid of the differential element as lever arm
  (not the coordinate describing the boundary of the area)





Modified Equations

$$\bar{x} = \frac{\int x_c dA}{A} \qquad \bar{y} = \frac{\int y_c dA}{A} \qquad \bar{z} = \frac{\int z_c dA}{A}$$

$$\bar{x} = \frac{\int x_c dV}{V} \qquad \bar{y} = \frac{\int y_c dV}{V} \qquad \bar{z} = \frac{\int z_c dV}{V}$$

Center of Mass and Centroids

Centroids of Lines, Areas, and Volumes

Guidelines for Choice of Elements for Integration

1. Order of Element Selected for Integration
2. Continuity
3. Discarding Higher Order Terms
4. Choice of Coordinates
5. Centroidal Coordinate of Differential Elements

$$\bar{x} = \frac{\int x\,dL}{L} \qquad \bar{y} = \frac{\int y\,dL}{L} \qquad \bar{z} = \frac{\int z\,dL}{L}$$

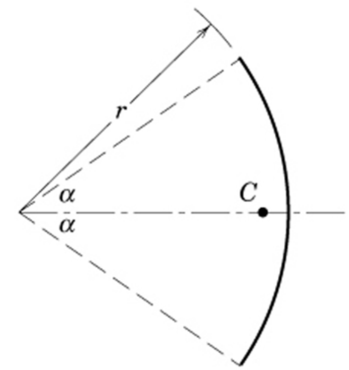
$$\bar{x} = \frac{\int x_c\,dA}{A} \qquad \bar{y} = \frac{\int y_c\,dA}{A} \qquad \bar{z} = \frac{\int z_c\,dA}{A}$$

$$\bar{x} = \frac{\int x_c\,dV}{V} \qquad \bar{y} = \frac{\int y_c\,dV}{V} \qquad \bar{z} = \frac{\int z_c\,dV}{V}$$

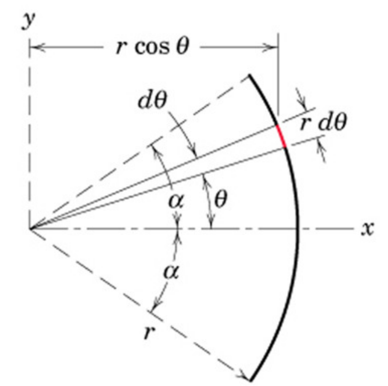
Center of Mass and Centroids

Examples: Centroids

Locate the centroid of the circular arc



Solution: Polar coordinate system is better
Since the figure is symmetric: centroid lies on the x axis



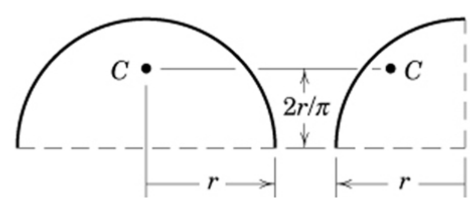
Differential element of arc has length $dL = rd\Theta$
Total length of arc: $L = 2\alpha r$
x-coordinate of the centroid of differential element: $x = rcos\Theta$

$$\bar{x} = \frac{\int x dL}{L} \quad \bar{y} = \frac{\int y dL}{L} \quad \bar{z} = \frac{\int z dL}{L}$$

$$\mathrm{L}\bar{x} = \int x dL \qquad 2\alpha r\bar{x} = \int_{-\alpha}^{\alpha} (rcos\theta) r d\theta$$

$$2\alpha r\bar{x} = 2r^2 sin\alpha$$

$$\bar{x} = \frac{rsin\alpha}{\alpha}$$

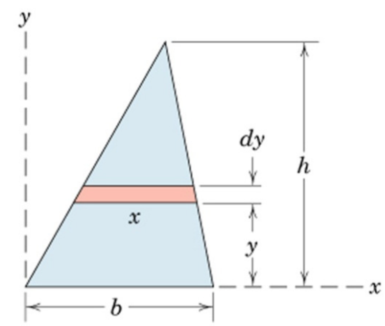


For a semi-circular arc: $2\alpha = \pi \rightarrow$ centroid lies at $2r/\pi$

Center of Mass and Centroids

Examples: Centroids

Locate the centroid of the triangle along h from the base



Solution:

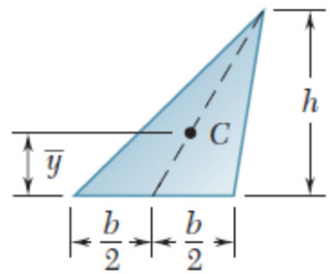
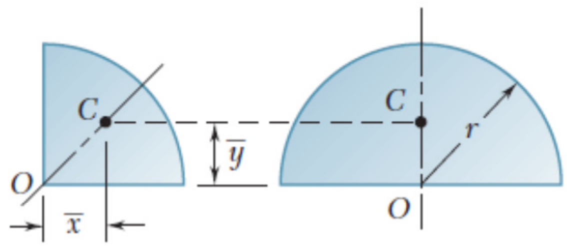
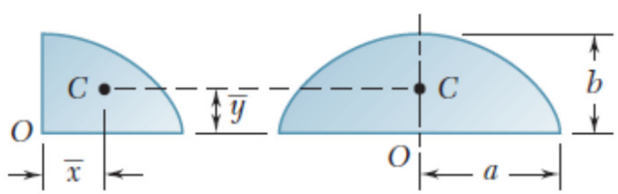
$dA = xdy \qquad \dfrac{x}{(h-y)} = \dfrac{b}{h}$

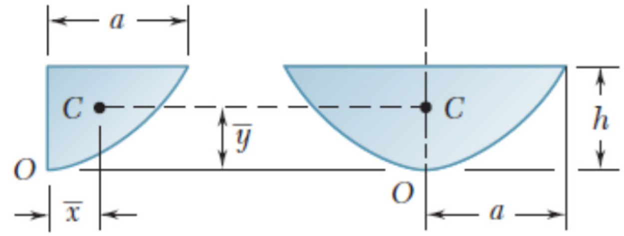
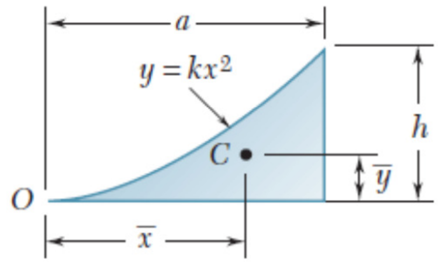
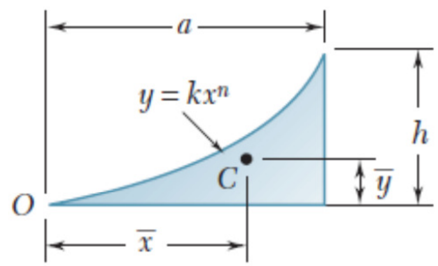
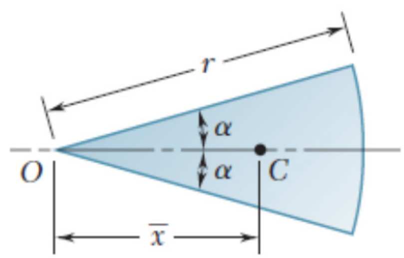
Total Area $A = \frac{1}{2}bh \qquad y = y_c$

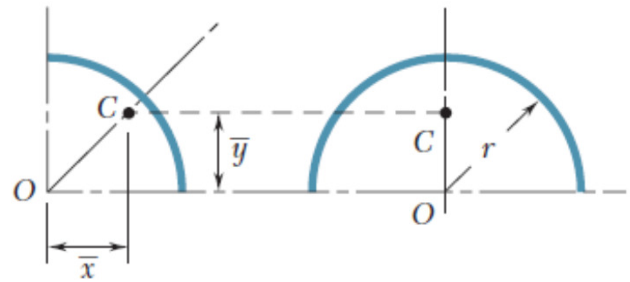
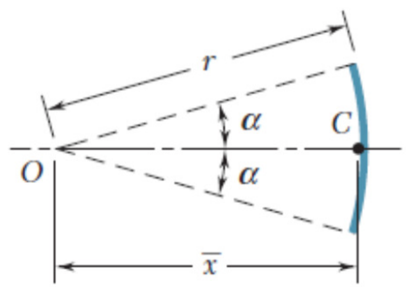
$$\bar{x} = \frac{\int x_c dA}{A} \quad \bar{y} = \frac{\int y_c dA}{A} \quad \bar{z} = \frac{\int z_c dA}{A}$$

$$A\bar{y} = \int y_c dA \quad \Rightarrow \frac{bh}{2}\bar{y} = \int_0^h y \frac{b(h-y)}{y} dy = \frac{bh^2}{6}$$

$$\boxed{\bar{y} = \frac{h}{3}}$$

Shape		$\overline{x}$	$\overline{y}$	Area
Triangular area			$\frac{h}{3}$	$\frac{bh}{2}$
Quarter-circular area		$\frac{4r}{3\pi}$	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{4}$
Semicircular area		0	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{2}$
Quarter-elliptical area		$\frac{4a}{3\pi}$	$\frac{4b}{3\pi}$	$\frac{\pi ab}{4}$
Semielliptical area		0	$\frac{4b}{3\pi}$	$\frac{\pi ab}{2}$

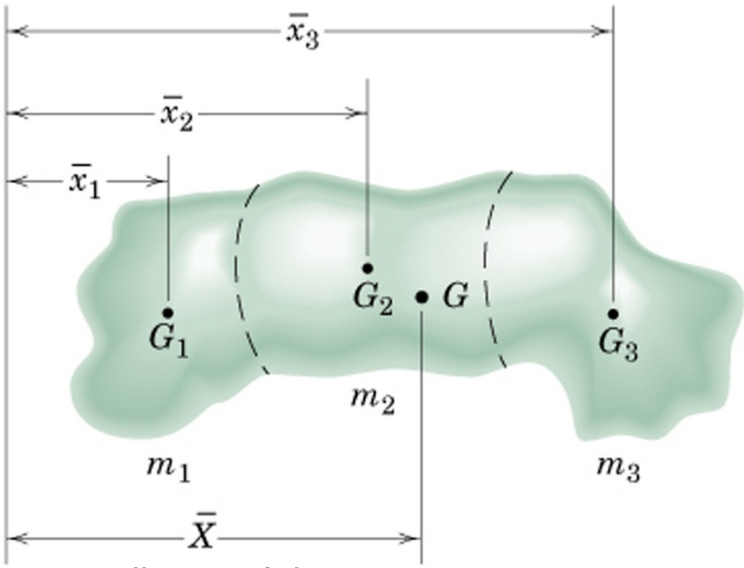
Semiparabolic area		$\frac{3a}{8}$	$\frac{3h}{5}$	$\frac{2ah}{3}$
Parabolic area		0	$\frac{3h}{5}$	$\frac{4ah}{3}$
Parabolic spandrel		$\frac{3a}{4}$	$\frac{3h}{10}$	$\frac{ah}{3}$
General spandrel		$\frac{n+1}{n+2}a$	$\frac{n+1}{4n+2}h$	$\frac{ah}{n+1}$
Circular sector		$\frac{2r \sin \alpha}{3\alpha}$	0	αr^2

Shape		$\overline{x}$	$\overline{y}$	Length
Quarter-circular arc		$\dfrac{2r}{\pi}$	$\dfrac{2r}{\pi}$	$\dfrac{\pi r}{2}$
Semicircular arc		0	$\dfrac{2r}{\pi}$	πr
Arc of circle		$\dfrac{r \sin \alpha}{\alpha}$	0	$2\alpha r$

Center of Mass and Centroids

Composite Bodies and Figures

Divide bodies or figures into several parts such that their mass centers can be conveniently determined
→ Use Principle of Moment for all finite elements of the body



x-coordinate of the center of mass of the whole

$$(m_1 + m_2 + m_3)\bar{X} = m_1\bar{x}_1 + m_2\bar{x}_2 + m_3\bar{x}_3$$

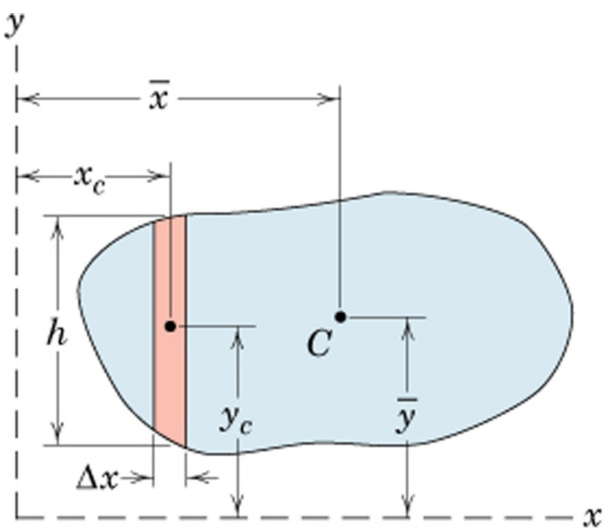
Mass Center Coordinates can be written as:

$$\bar{X} = \frac{\sum m\bar{x}}{\sum m} \quad \bar{Y} = \frac{\sum m\bar{y}}{\sum m} \quad \bar{Z} = \frac{\sum m\bar{z}}{\sum m}$$

m's can be replaced by *L*'s, *A*'s, and *V*'s for lines, areas, and volumes

Integration vs Appx Summation: Irregular Area

In some cases, the boundaries of an area or volume might not be expressible mathematically or in terms of simple geometrical shapes → Appx Summation may be used instead of integration



Divide the area into several strips

Area of each strip = $h\Delta x$

Moment of this area about x- and y-axis

= $(h\Delta x)y_c$ *and* $(h\Delta x)x_c$

→ Sum of moments for all strips divided by the total area will give corresponding coordinate of the centroid

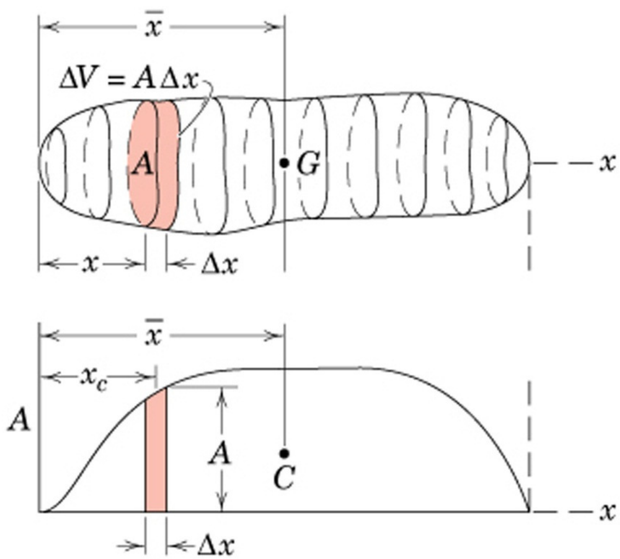
$$\bar{x} = \frac{\sum A x_c}{\sum A} \qquad \bar{y} = \frac{\sum A y_c}{\sum A}$$

Accuracy may be improved by reducing the width of the strip

Center of Mass and Centroids: Composite Bodies and Figures

Integration vs Appx Summation: Irregular Volume

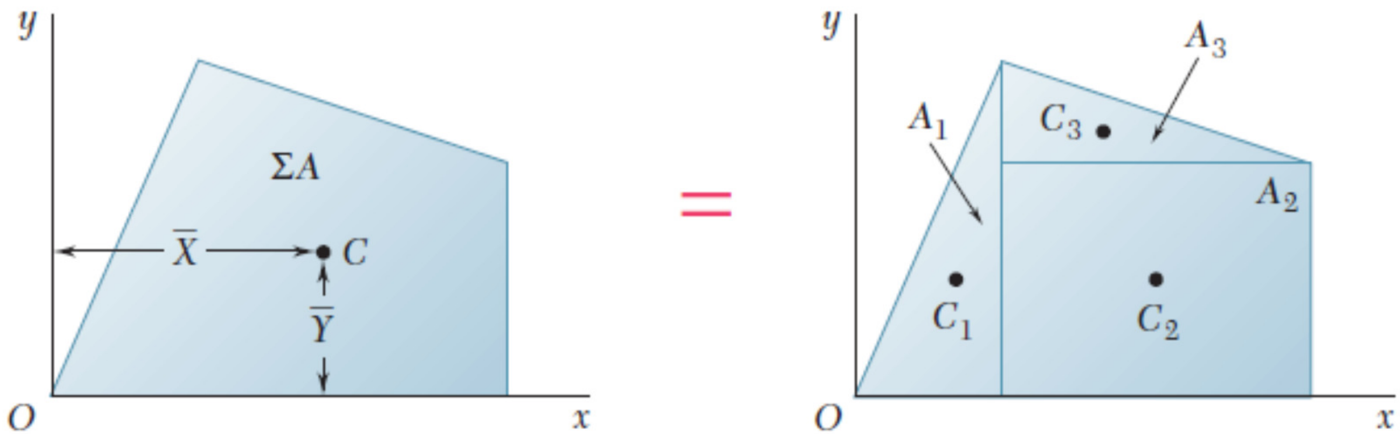
Reduce the problem to one of locating the centroid of area
→ Appx Summation may be used instead of integration



Divide the area into several strips
Volume of each strip = $A\Delta x$
Plot all such A against x.
→ Area under the plotted curve represents volume of whole body and the x-coordinate of the centroid of the area under the curve is given by:

$$\bar{x} = \frac{\sum (A\Delta x) x_c}{\sum A\Delta x} \Rightarrow \bar{x} = \frac{\sum V x_c}{\sum V}$$

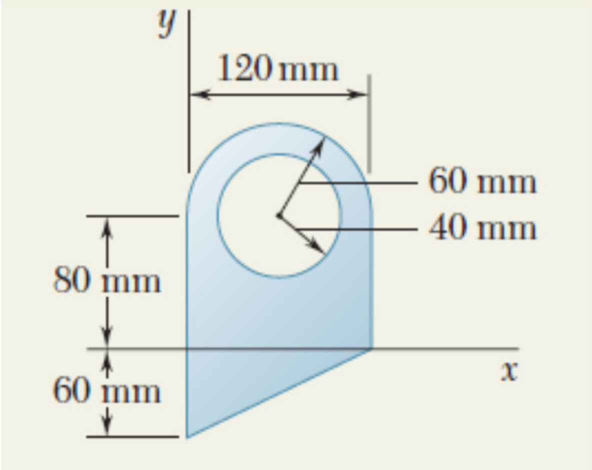
Accuracy may be improved by reducing the width of the strip

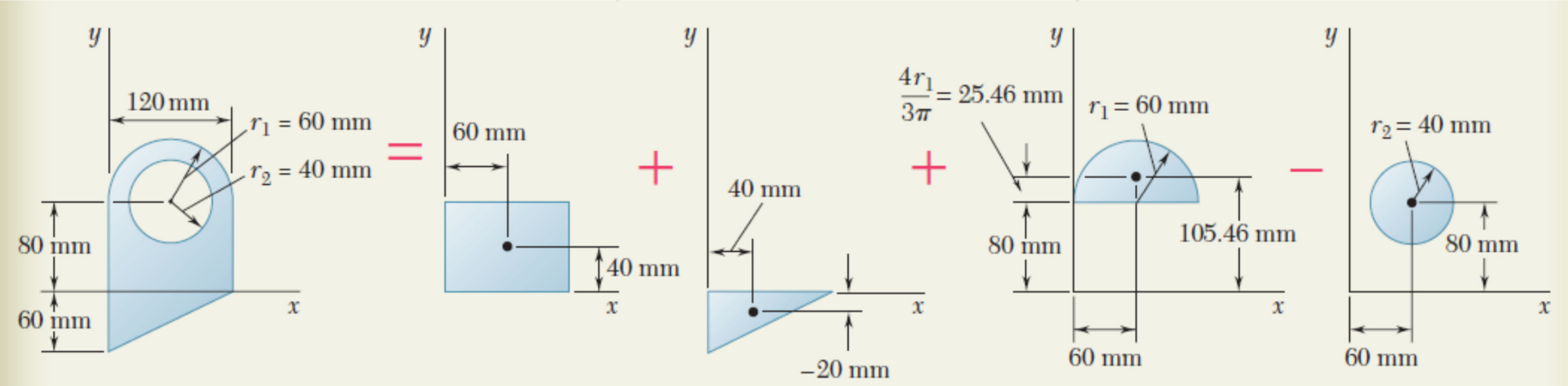


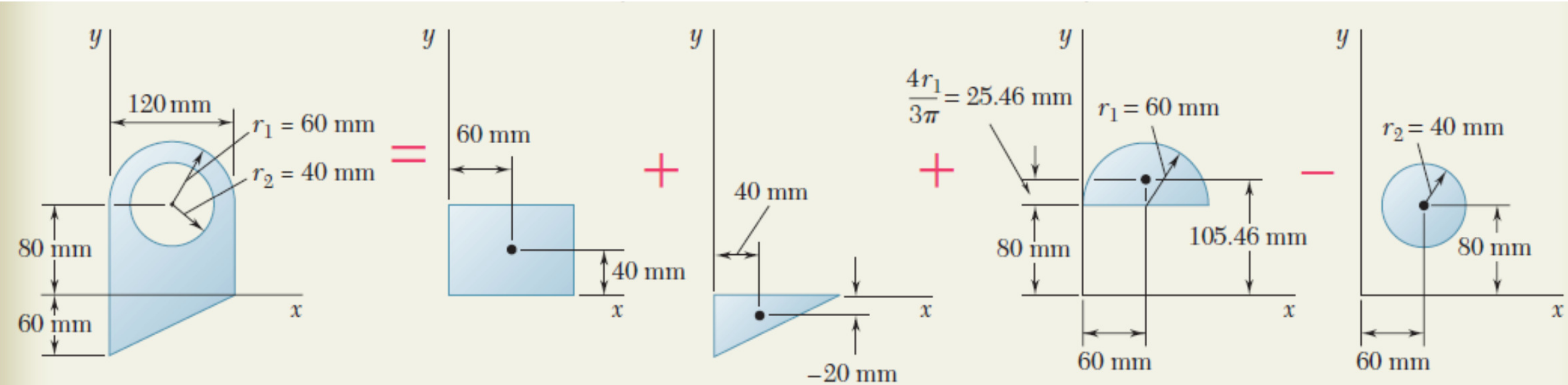
$$Q_y = \overline{X}\,\Sigma A = \Sigma\,\overline{x}\,A$$

$$Q_x = \overline{Y}\,\Sigma A = \Sigma\,\overline{y}\,A$$

For the plane area shown, determine (*a*) the first moments with respect to the x and y axes, (*b*) the location of the centroid.







Component	A, mm^2	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}A$, mm^3	$\bar{y}A$, mm^3
Rectangle	$(120)(80) = 9.6 \times 10^3$	60	40	$+576 \times 10^3$	$+384 \times 10^3$
Triangle	$\frac{1}{2}(120)(60) = 3.6 \times 10^3$	40	−20	$+144 \times 10^3$	-72×10^3
Semicircle	$\frac{1}{2}\pi(60)^2 = 5.655 \times 10^3$	60	105.46	$+339.3 \times 10^3$	$+596.4 \times 10^3$
Circle	$-\pi(40)^2 = -5.027 \times 10^3$	60	80	-301.6×10^3	-402.2×10^3
	$\Sigma A = 13.828 \times 10^3$			$\Sigma\bar{x}A = +757.7 \times 10^3$	$\Sigma\bar{y}A = +506.2 \times 10^3$

First Moments of the Area.

$$Q_x = \Sigma\bar{y}A = 506.2 \times 10^3\ \text{mm}^3$$
$$Q_y = \Sigma\bar{x}A = 757.7 \times 10^3\ \text{mm}^3$$

Location of Centroid.

$$\bar{X}\Sigma A = \Sigma\bar{x}A: \qquad \bar{X}(13.828 \times 10^3\ \text{mm}^2) = 757.7 \times 10^3\ \text{mm}^3$$
$$\bar{X} = 54.8\ \text{mm}$$

$$\bar{Y}\Sigma A = \Sigma\bar{y}A: \qquad \bar{Y}(13.828 \times 10^3\ \text{mm}^2) = 506.2 \times 10^3\ \text{mm}^3$$
$$\bar{Y} = 36.6\ \text{mm}$$

Center of Mass and Centroids: Composite Bodies and Figures

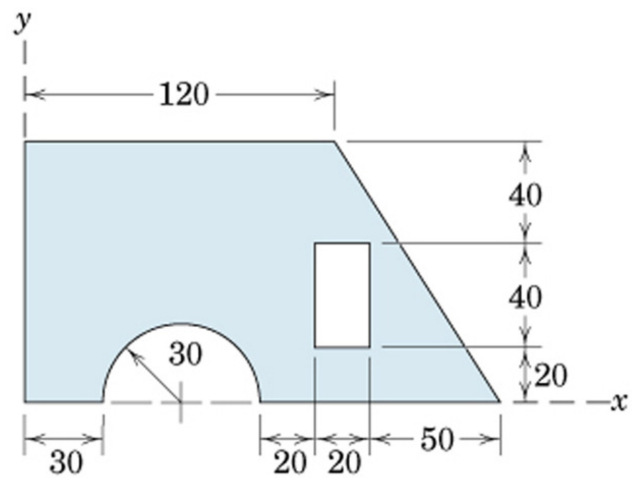
Example:
Locate the centroid of the shaded area

Solution: Divide the area into four elementary shapes: Total Area = $A_1 + A_2 - A_3 - A_4$

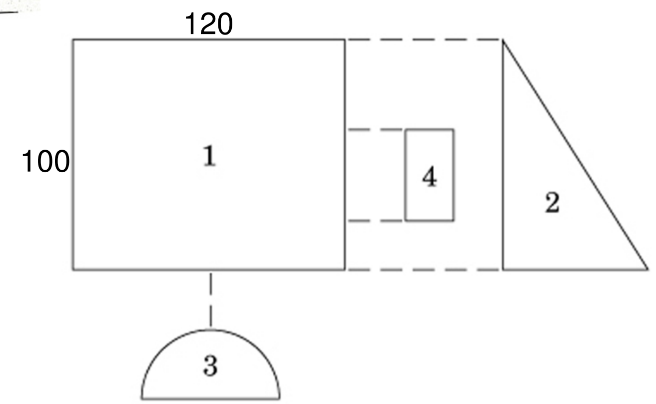
PART	A mm^2	$\bar{x}$ mm	$\bar{y}$ mm	$\bar{x}A$ mm^3	$\bar{y}A$ mm^3
1	12 000	60	50	720 000	600 000
2	3000	140	100/3	420 000	100 000
3	−1414	60	12.73	−84 800	−18 000
4	−800	120	40	−96 000	−32 000
TOTALS	12 790			959 000	650 000

$$\left[\bar{X} = \frac{\Sigma A\bar{x}}{\Sigma A}\right] \qquad \bar{X} = \frac{959\ 000}{12\ 790} = 75.0 \text{ mm}$$

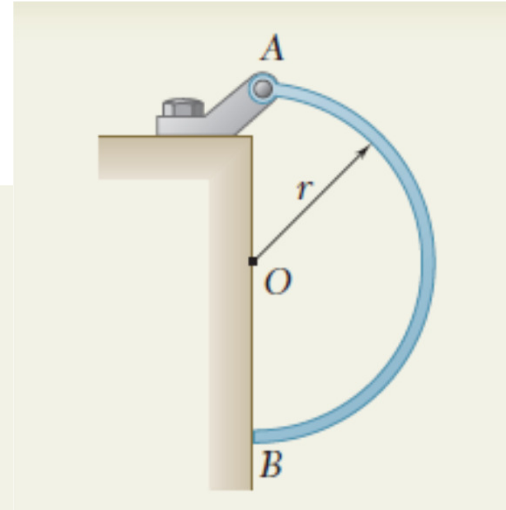
$$\left[\bar{Y} = \frac{\Sigma A\bar{y}}{\Sigma A}\right] \qquad \bar{Y} = \frac{650\ 000}{12\ 790} = 50.8 \text{ mm}$$



Dimensions in millimeters



A uniform semicircular rod of weight W and radius r is attached to a pin at A and rests against a frictionless surface at B. Determine the reactions at A and B.



$$+\circlearrowleft \Sigma M_A = 0: \qquad B(2r) - W\left(\frac{2r}{\pi}\right) = 0$$

$$B = +\frac{W}{\pi}$$

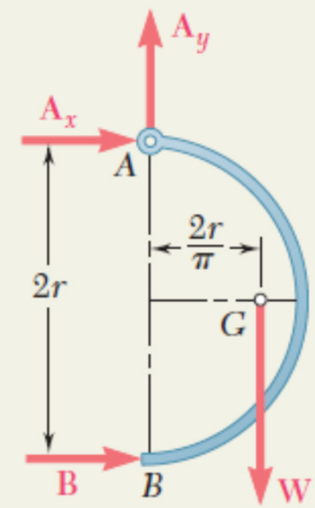
$$\overset{+}{\rightarrow} \Sigma F_x = 0: \qquad A_x + B = 0$$

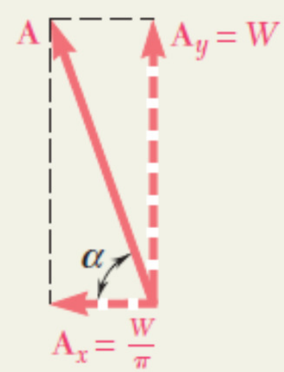
$$A_x = -B = -\frac{W}{\pi} \qquad \mathbf{A}_x = \frac{W}{\pi} \leftarrow$$

$$+\uparrow \Sigma F_y = 0: \qquad A_y - W = 0 \qquad \mathbf{A}_y = W \uparrow$$

$$A = \left[W^2 + \left(\frac{W}{\pi}\right)^2\right]^{1/2}$$

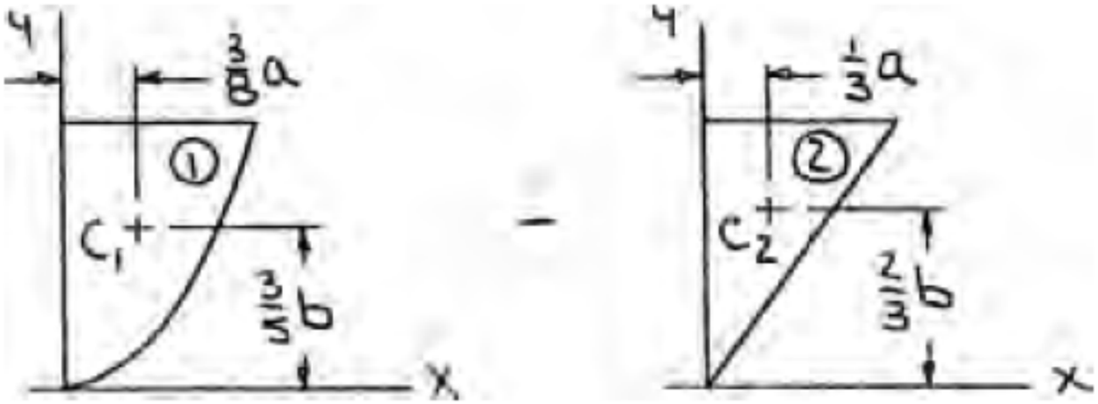
$$\tan\alpha = \frac{W}{W/\pi} = \pi$$

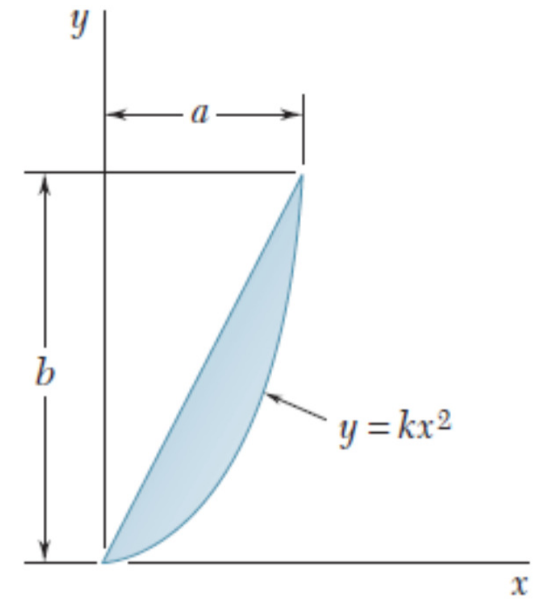




$$\mathbf{A} = 1.049W \;\measuredangle 72.3^\circ \qquad \mathbf{B} = 0.318W \rightarrow$$

For the area shown, determine the ratio a/b for which $\bar{x} = \bar{y}$





	A	$\bar{x}$	$\bar{y}$	$\bar{x}A$	$\bar{y}A$
1	$\frac{2}{3}ab$	$\frac{3}{8}a$	$\frac{3}{5}b$	$\frac{a^2b}{4}$	$\frac{2ab^2}{5}$
2	$-\frac{1}{2}ab$	$\frac{1}{3}a$	$\frac{2}{3}b$	$-\frac{a^2b}{6}$	$-\frac{ab^2}{3}$
Σ	$\frac{1}{6}ab$			$\frac{a^2b}{12}$	$\frac{ab^2}{15}$

Then $$\bar{X}\,\Sigma A = \Sigma\,\bar{x}A$$

$$\bar{X}\left(\frac{1}{6}ab\right) = \frac{a^2 b}{12}$$

or $$\bar{X} = \frac{1}{2}a$$

$$\bar{Y}\,\Sigma A = \Sigma\,\bar{y}A$$

$$\bar{Y}\left(\frac{1}{6}ab\right) = \frac{ab^2}{15}$$

or $$\bar{Y} = \frac{2}{5}b$$

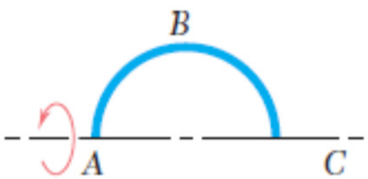
Now $$\bar{X} = \bar{Y} \Rightarrow \quad \frac{1}{2}a = \frac{2}{5}b \qquad \text{or} \quad \frac{a}{b} = \frac{4}{5} \blacktriangleleft$$

Center of Mass and Centroids

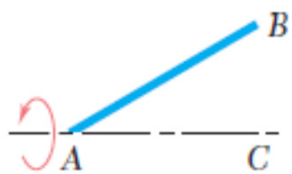
Theorems of Pappus: Areas and Volumes of Revolution

Method for calculating surface area generated by revolving a plane curve about a non-intersecting axis in the plane of the curve

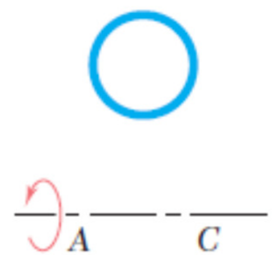
Method for calculating volume generated by revolving an area about a non intersecting axis in the plane of the area



Sphere

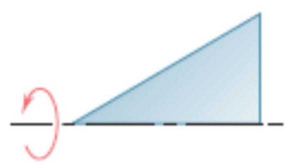


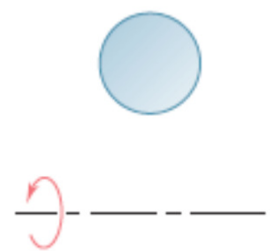
Cone



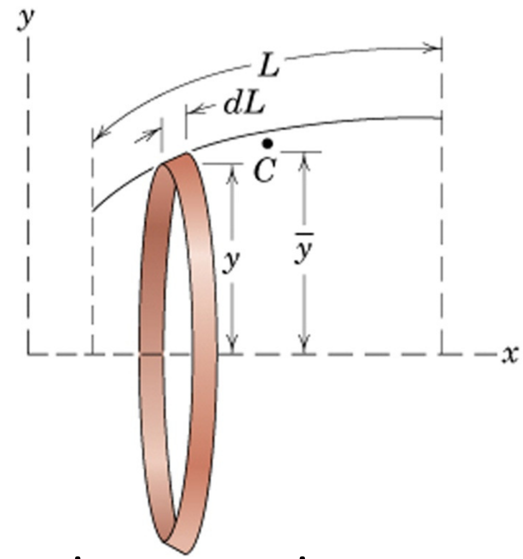
Torus


Sphere


Cone


Torus

Center of Mass and Centroids



Surface Area

Area of the ring element: circumference times *dL*
$dA = 2\pi y\, dL$

Total area, $$A = 2\pi \int y dL \qquad \bar{y}L = \int y dL$$

$$A = 2\pi\bar{y}L$$

The area of a surface of revolution is equal to the length of the generating curve times the distance traveled by the centroid of the curve while the surface is being generated

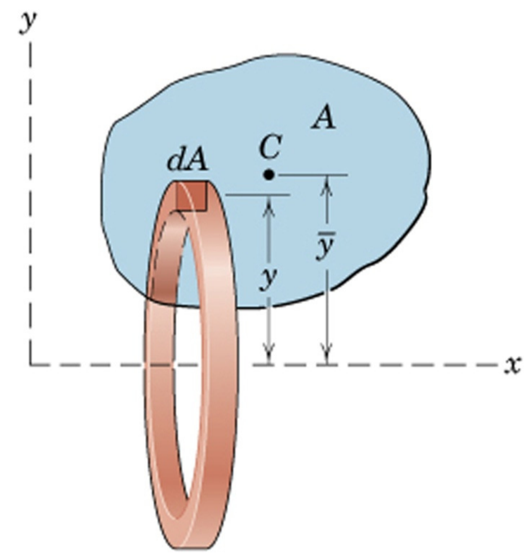
This is Theorems 1 of Pappus

If area is revolved through an angle $\theta<2\pi$, θ in radians

$$A = \theta\bar{y}L$$

Theorems of Pappus can also be used to determine centroid of plane curves if area created by revolving these figures @ a non-intersecting axis is known

Center of Mass and Centroids



Volume

Volume of the ring element: circumference times *dA*

$dV = 2\pi y\, dA$

Total Volume, $V = 2\pi \int y dA$ $\bar{y}A = \int y dA$

$$V = 2\pi\bar{y}A$$

The volume of a body of revolution is equal to the length of the generating area times the distance traveled by the centroid of the area while the body is being generated

This is Theorems 2 of Pappus

If area is revolved through an angle $\theta<2\pi$, θ in radians

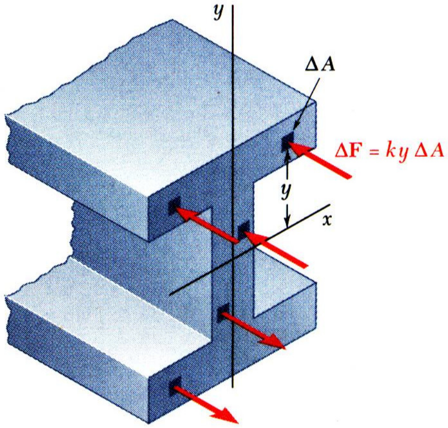
$$V = \theta\bar{y}A$$

Area Moments of Inertia

- Previously considered distributed forces which were proportional to the area or volume over which they act.
  - The resultant was obtained by summing or integrating over the areas or volumes.
  - The moment of the resultant about any axis was determined by computing the first moments of the areas or volumes about that axis.

- Will now consider forces which are proportional to the area or volume over which they act but also vary linearly with distance from a given axis.
  - the magnitude of the resultant depends on the first moment of the force distribution with respect to the axis.
  - The point of application of the resultant depends on the second moment of the distribution with respect to the axis.

Area Moments of Inertia



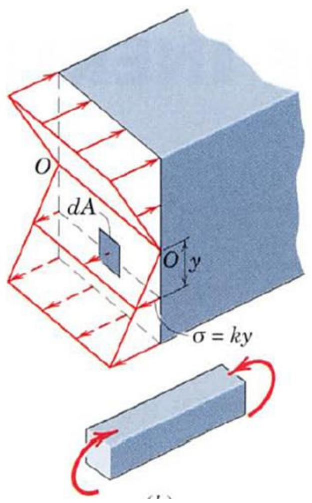
- Consider distributed forces $\Delta\vec{F}$ whose magnitudes are proportional to the elemental areas ΔA on which they act and also vary linearly with the distance of ΔA from a given axis.

- Example: Consider a beam subjected to pure bending. Internal forces vary linearly with distance from the neutral axis which passes through the section centroid.

$$\Delta\vec{F} = ky\Delta A$$

$$R = k\int y\,dA = 0 \qquad \int y\,dA = Q_x = \text{first moment}$$

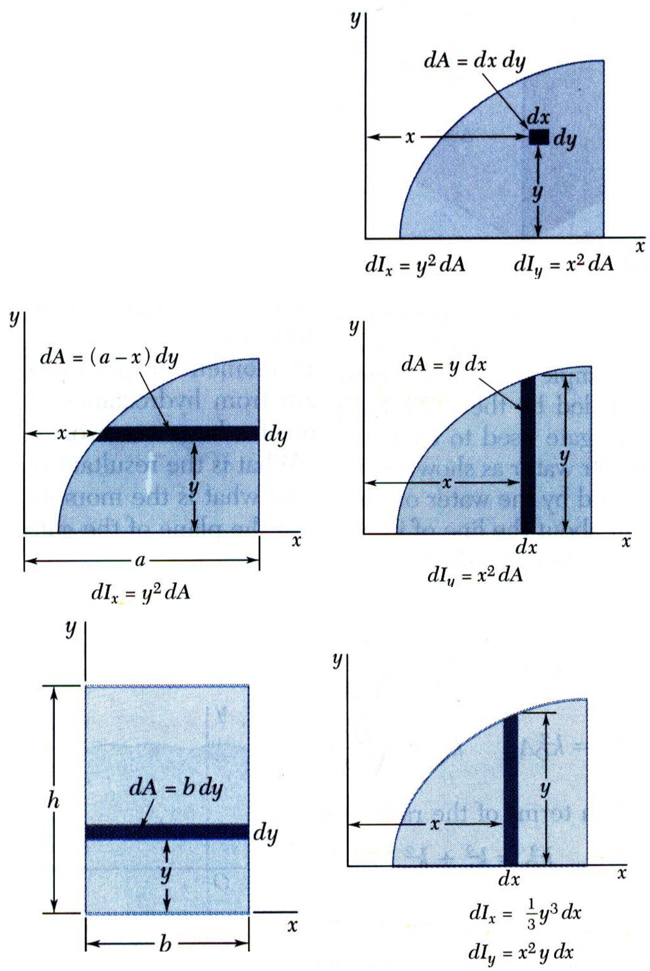
$$M = k\int y^2 dA \qquad \int y^2 dA = \text{second moment}$$



First Moment of the whole section about the x-axis = $\bar{y}A = 0$ since the centroid of the section lies on the x-axis.

Second Moment or the Moment of Inertia of the beam section about x-axis is denoted by I_x and has units of $(\text{length})^4$ (never –ve)

Area Moments of Inertia by Integration



- *Second moments* or *moments of inertia* of an area with respect to the *x* and *y* axes,

$$I_x = \int y^2 dA \qquad I_y = \int x^2 dA$$

- Evaluation of the integrals is simplified by choosing dA to be a thin strip parallel to one of the coordinate axes.

- For a rectangular area,

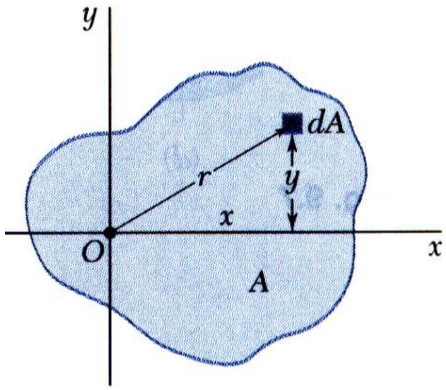
$$I_x = \int y^2 dA = \int_0^h y^2 b\,dy = \tfrac{1}{3} b h^3$$

- The formula for rectangular areas may also be applied to strips parallel to the axes,

$$dI_x = \tfrac{1}{3} y^3 dx \qquad dI_y = x^2 dA = x^2 y\,dx$$

Area Moments of Inertia

Polar Moment of Inertia



- The *polar moment of inertia* is an important parameter in problems involving torsion of cylindrical shafts and rotations of slabs.

$$J_0 = I_z = \int r^2 dA$$

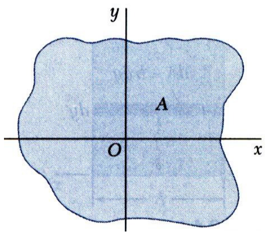
- The polar moment of inertia is related to the rectangular moments of inertia,

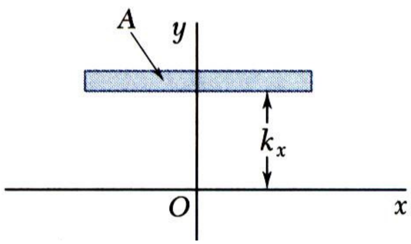
$$J_0 = I_z = \int r^2 dA = \int \left(x^2 + y^2\right) dA = \int x^2 dA + \int y^2 dA$$
$$= I_y + I_x$$

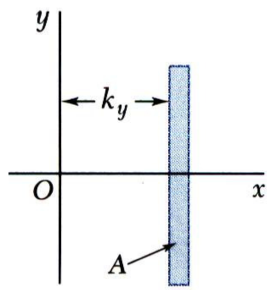
Moment of Inertia of an area is purely a mathematical property of the area and in itself has no physical significance.

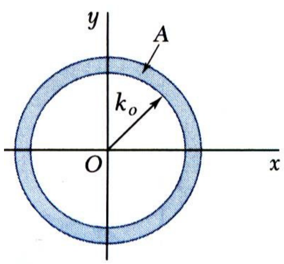
Area Moments of Inertia

Radius of Gyration of an Area









- Consider area A with moment of inertia I_x. Imagine that the area is concentrated in a thin strip parallel to the x axis with equivalent I_x.

$$I_x = k_x^2 A \qquad k_x = \sqrt{\frac{I_x}{A}}$$

$k_x =$ *radius of gyration* with respect to the x axis

- Similarly,

$$I_y = k_y^2 A \qquad k_y = \sqrt{\frac{I_y}{A}}$$

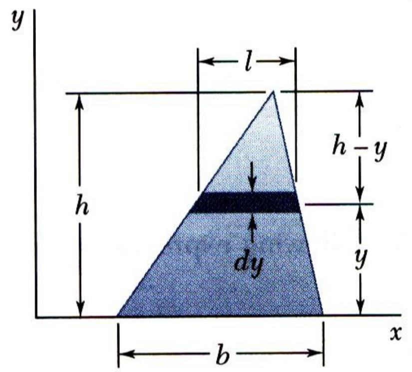
$$J_O = I_z = k_O^2 A = k_z^2 A \qquad k_O = k_z = \sqrt{\frac{J_O}{A}}$$

$$k_O^2 = k_z^2 = k_x^2 + k_y^2$$

Radius of Gyration, *k* is a measure of distribution of area from a reference axis
Radius of Gyration is different from centroidal distances

Area Moments of Inertia

Example: Determine the moment of inertia of a triangle with respect to its base.



SOLUTION:

- A differential strip parallel to the x axis is chosen for dA.

$$dI_x = y^2 dA \qquad dA = l\,dy$$

- For similar triangles,

$$\frac{l}{b} = \frac{h-y}{h} \qquad l = b\frac{h-y}{h} \qquad dA = b\frac{h-y}{h}dy$$

- Integrating dI_x from $y = 0$ to $y = h$,

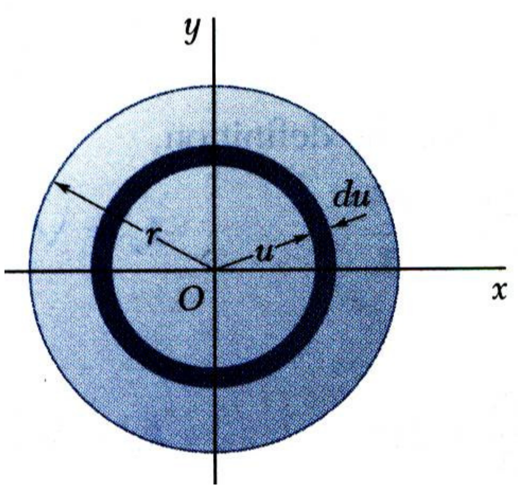
$$I_x = \int y^2 dA = \int_0^h y^2 b\frac{h-y}{h}dy = \frac{b}{h}\int_0^h \left(hy^2 - y^3\right)dy$$

$$= \frac{b}{h}\left[h\frac{y^3}{3} - \frac{y^4}{4}\right]_0^h$$

$$\boxed{I_x = \frac{bh^3}{12}}$$

Area Moments of Inertia

Example: a) Determine the centroidal polar moment of inertia of a circular area by direct integration.
b) Using the result of part a, determine the moment of inertia of a circular area with respect to a diameter.



SOLUTION:

- An annular differential area element is chosen,

$$dJ_O = u^2 dA \qquad dA = 2\pi u\, du$$

$$J_O = \int dJ_O = \int_0^r u^2 (2\pi u\, du) = 2\pi \int_0^r u^3 du$$

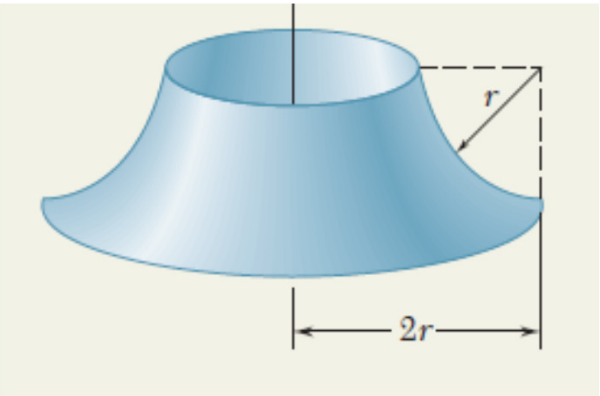
$$\boxed{J_O = \frac{\pi}{2} r^4}$$

- From symmetry, $I_x = I_y$,

$$J_O = I_x + I_y = 2I_x \qquad \frac{\pi}{2} r^4 = 2I_x$$

$$\boxed{I_{diameter} = I_x = \frac{\pi}{4} r^4}$$

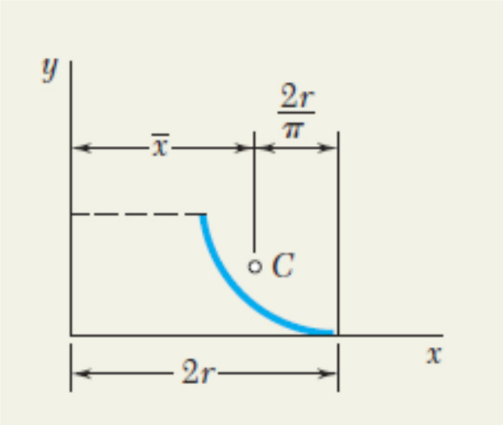
Determine the area of the surface of revolution shown



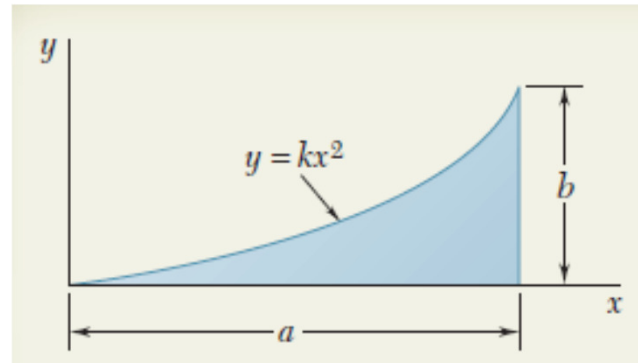
$$\bar{x} = 2r - \frac{2r}{\pi} = 2r\left(1 - \frac{1}{\pi}\right)$$

$$A = 2\pi\bar{x}L = 2\pi\left[2r\left(1 - \frac{1}{\pi}\right)\right]\left(\frac{\pi r}{2}\right)$$

$$A = 2\pi r^2(\pi - 1)$$



(*a*) Determine the moment of inertia of the shaded area shown with respect to each of the coordinate axes. (Properties of this area were considered in Sample Prob. 5.4.) (*b*) Using the results of part *a*, determine the radius of gyration of the shaded area with respect to each of the coordinate axes.

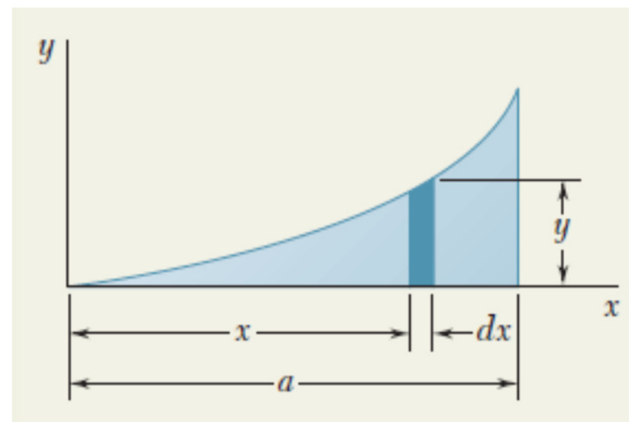


$$y = \frac{b}{a^2}x^2 \qquad A = \tfrac{1}{3}ab$$

$$dI_x = \tfrac{1}{3}y^3\,dx = \frac{1}{3}\left(\frac{b}{a^2}x^2\right)^3 dx = \frac{1}{3}\frac{b^3}{a^6}x^6\,dx$$

$$I_x = \int dI_x = \int_0^a \frac{1}{3}\frac{b^3}{a^6}x^6\,dx = \left[\frac{1}{3}\frac{b^3}{a^6}\frac{x^7}{7}\right]_0^a$$

$$I_x = \frac{ab^3}{21}$$



$$dI_y = x^2\,dA = x^2(y\,dx) = x^2\left(\frac{b}{a^2}x^2\right)dx = \frac{b}{a^2}x^4\,dx$$

$$I_y = \int dI_y = \int_0^a \frac{b}{a^2}x^4\,dx = \left[\frac{b}{a^2}\frac{x^5}{5}\right]_0^a$$

$$I_y = \frac{a^3 b}{5}$$

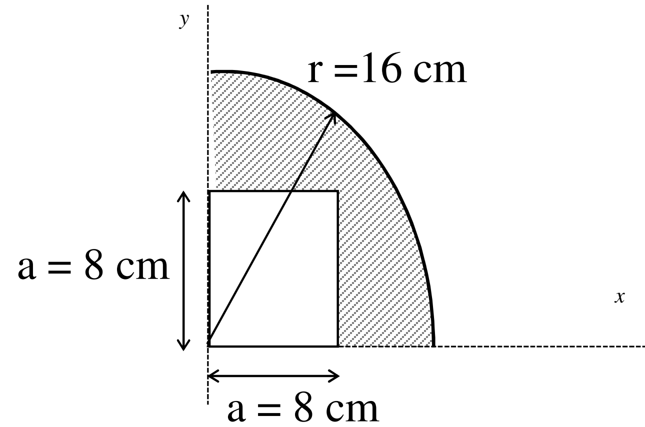
Radii of Gyration k_x and k_y. We have, by definition,

$$k_x^2 = \frac{I_x}{A} = \frac{ab^3/21}{ab/3} = \frac{b^2}{7} \qquad k_x = \sqrt{\tfrac{1}{7}}b$$

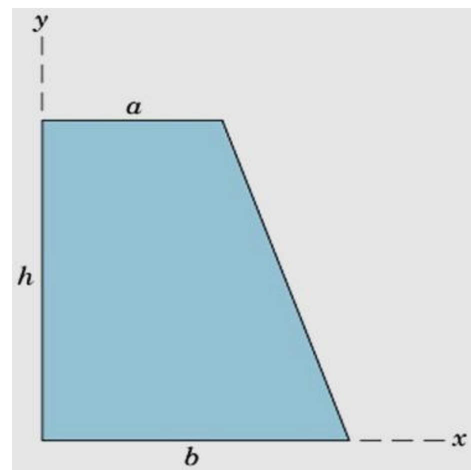
and

$$k_y^2 = \frac{I_y}{A} = \frac{a^3b/5}{ab/3} = \tfrac{3}{5}a^2 \qquad k_y = \sqrt{\tfrac{3}{5}}a$$

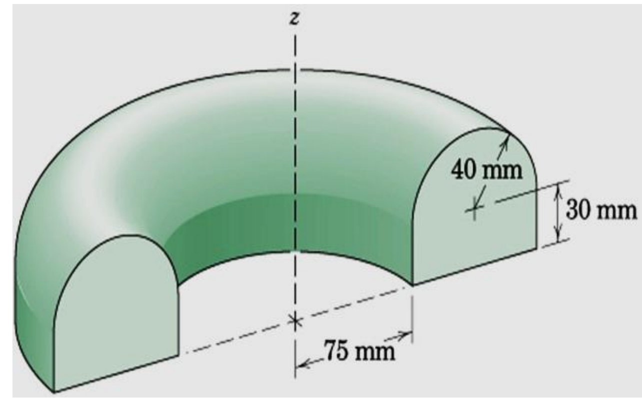
Q. No. 1 Locate the centroid of the plane shaded area shown below.



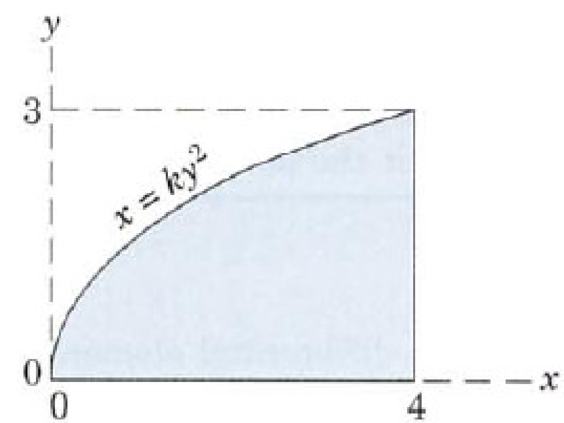
Q. No. 2 Determine *x*- and *y*-coordinates of the centroid of the trapezoidal area



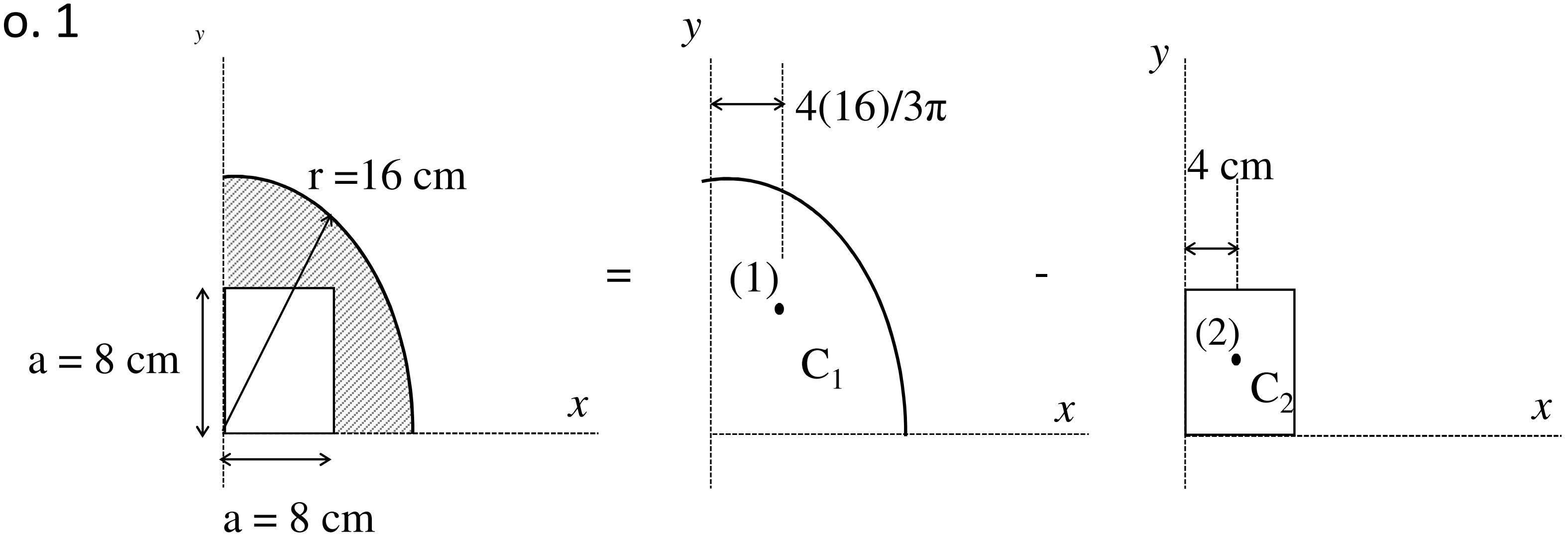
Q. No. 3 Determine the volume V and total surface area A of the solid generated by revolving the area shown through 180º about the z-axis.



Q. No. 4 Determine moment of inertia of the area under the parabola about x-axis. Solve by using (a) a horizontal strip of area and (b) a vertical strip of area.



Solution of Q. No. 1



	A (cm^2)	$\bar{x}$, cm	$\bar{x}$A, cm^3
1	$\frac{\pi(16^2)}{4}$ = 201.06	$\frac{4(16)}{3\pi}$ = 6.7906	1365.32
2	-(8)(8) = - 64	4	-256
∑	137.06		1109.32

Then $\bar{X} = \frac{\sum \bar{x}A}{\sum A} = \frac{1109.32}{137.06} = 8.09$ cm

$\bar{Y} = 8.09$ cm ($\bar{Y} = \bar{X}$ by symmetry)

Solution of Q. No. 2

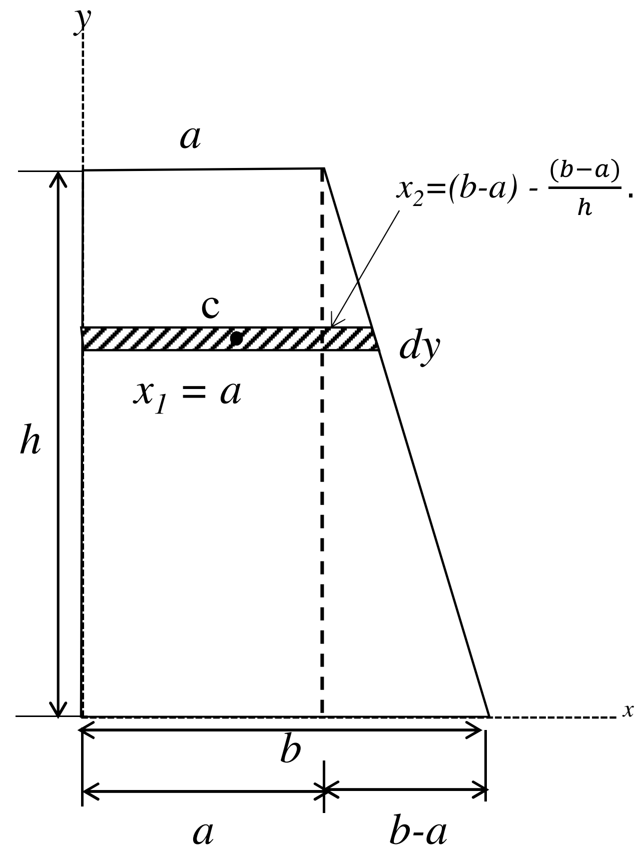
Dividing the trapezoid into a rectangle of dimensions (a × h) and a triangle of base width (b-a).

Total $x = x_1 + x_2 = a + (b\text{-}a) - \frac{(b-a)}{h}.y$

$$x = \left(\frac{a-b}{h}\right)y + b$$

$$dA = xdy = \left[\left(\frac{a-b}{h}\right)y + b\right]dy$$

Total area $A = \left(\frac{a+b}{2}\right)h = \frac{h}{2}(a+b)$



Centroid of differential element $x_c = \frac{x}{2}; y_c = y$

$$\int x_c\, dA = \int_0^h \frac{x}{2}\left[\left(\frac{a-b}{h}\right)y + b\right]dy$$

$$= \frac{1}{2}\int_0^h [\left(\frac{a-b}{h}\right)y + b]^2 dy$$

$$\int x_c dA = \frac{1}{2}\int_0^h [(\frac{a-b}{h})^2 y^2 + 2b\left(\frac{a-b}{h}\right)y + b^2]dy$$

$$= \frac{1}{2}[(\frac{a-b}{h})^2\frac{y^3}{3} + 2b\left(\frac{a-b}{h}\right)\frac{y^2}{2} + b^2y]_0^h$$

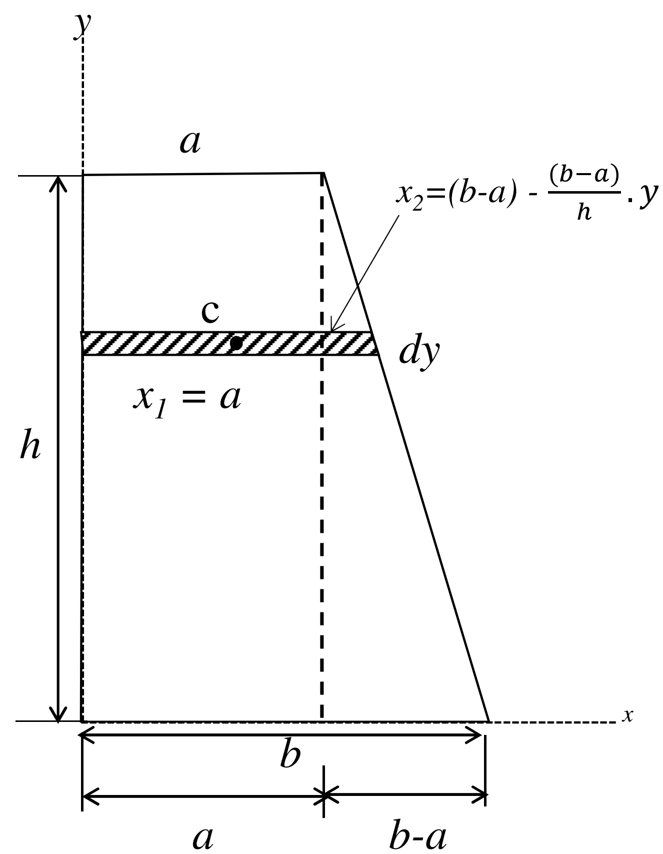
$$= \frac{h}{6}(a^2 + b^2 + ab)$$

$$\int y_c dA = \int_0^h y\left[\left(\frac{a-b}{h}\right)y + b\right]dy = \int_0^h [(\frac{a-b}{h})y^2 + by]\,dy$$

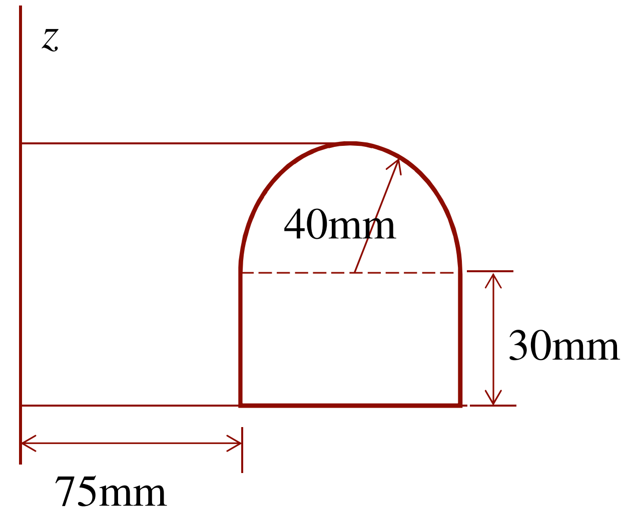
$$= [\left(\frac{a-b}{h}\right)\frac{y^3}{3} + b\,\frac{y^2}{2}]_0^h = \frac{h^2}{3}(a + \frac{b}{2})$$

$$\bar{x} = \frac{\int x_c dA}{A} = \frac{\frac{h}{6}(a^2 + b^2 + ab)}{\frac{h}{2}(a+b)} = \frac{a^2 + b^2 + ab}{3(a+b)}$$

$$\bar{y} = \frac{\int y_c dA}{A} = \frac{\frac{h^2}{3}(a + \frac{b}{2})}{\frac{h}{2}(a+b)} = \frac{h(2a+b)}{3(a+b)}$$



Solution of Q. No. 3

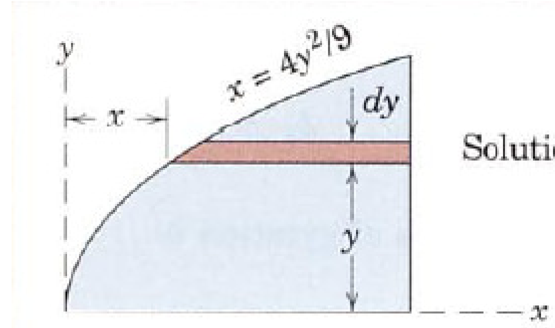


Let A_1 be the surface area excluding the end surfaces
And A_2 be the surface area of the end surfaces
Using Pappus theorem:
$A_1 = \pi\bar{r}L = \pi\,(75 + 40)\,(2\times30 + 80 + (\pi \times 40))$
$= 96000\ \text{mm}^2$

End areas $A_2 = 2\,(\frac{\pi}{2} \times 40^2 + 80 \times 30)$
$= 9830\ \text{mm}^2$
Total area $A = A_1 + A_2 = 105800\ \text{mm}^2$

Again using Pappus theorem for revolution of plane areas:
$V = \pi\bar{r}A = \pi\,(75 + 40)\,(30\times80 + \pi \times \frac{40^2}{2}) = 1775 \times 10^6\ \text{mm}^3$

Solution of Q. No. 4



Solution (a)

(a) Horizontal strip

$$I_x = \int y^2 dA \qquad I_x = \int_0^3 4y^2\left(1 - \frac{y^2}{9}\right) dy = 14.40$$

(b) Vertical strip

$$dI_x = \frac{1}{3}(dx)y^3 \qquad y = \frac{3\sqrt{x}}{2}$$

$$I_x = \frac{1}{3}\int_0^4 \left(\frac{3\sqrt{x}}{2}\right)^2 dx = 14.40$$