Arora's PTAS for the Euclidean TSP[1]

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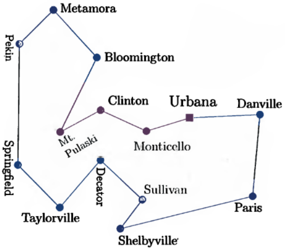
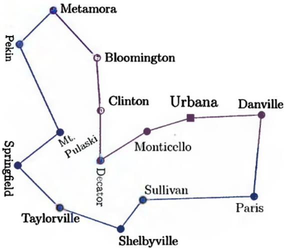
http://www.iitg.ac.in/rinkulu/

[1]most of the figures in this presentation are from the references mentioned

problem description

Given a set P of points in Euclidean plane, find a tour of minimum cost that visits all the points of P.





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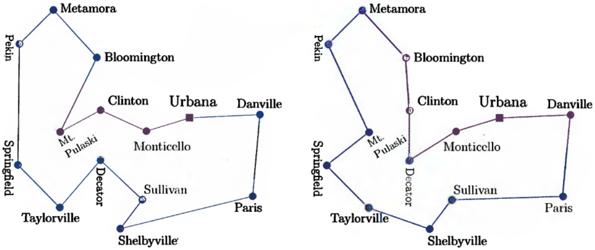

- known to be NP-hard
- objective of this talk: algo (PTAS) to compute an *approximate tour* whose *expected length* is $(1+\epsilon)\tau_{OPT}$ in $O(n(\frac{\lg n}{\epsilon})^{O(\frac{1}{\epsilon})})$ time

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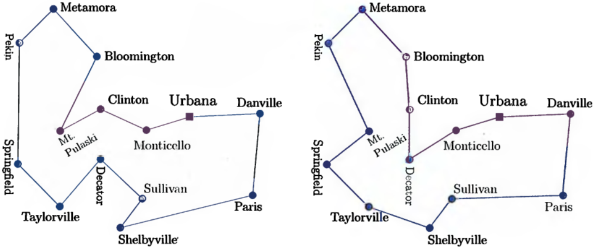

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for the convenience, we let the following:

$1 > \epsilon > \frac{1}{n}$

P is contained in $[\frac{1}{2}, \frac{1}{2}] \times [1, 1]$

$diam(P) \geq \frac{1}{4}$

Outline

1. algorithm

2. analysis

round the input points to grid points

- set up a grid with sidelength $G = \{\frac{1}{n\lceil\frac{32}{\epsilon}\rceil}(i,j) | i,j \text{ are integers}\}$

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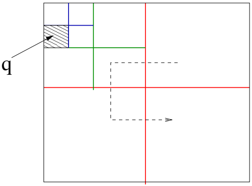
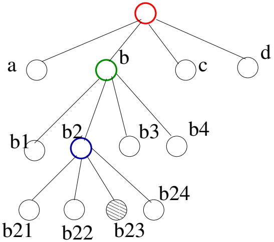
- set up a grid with sidelength $G = \{\frac{1}{n\lceil\frac{32}{\epsilon}\rceil}(i,j) | i,j \text{ are integers}\}$
- round (snap) each point in P to an arbitrary corner (grid point) of the grid cell in which it is located $\rightarrow$ the resultant point set is Q
  - this would guarantee the quadtree to be constructed for Q to have logarithmic depth

setting up a quadtree over Q





- consider the unit square with its southwest corner at $(\frac{1}{2}, \frac{1}{2})^2$ as the square of the root node

[2] later, this corner will be fixed at a point randomly chosen from the square with the southwest and northeast corners respectively at $(0, 0)$ and $(\frac{1}{2}, \frac{1}{2})$

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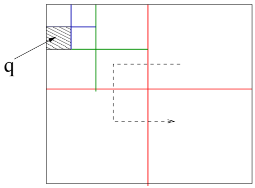

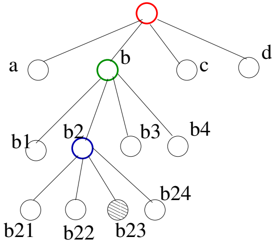

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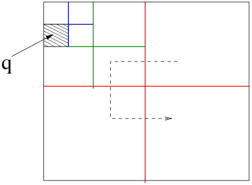

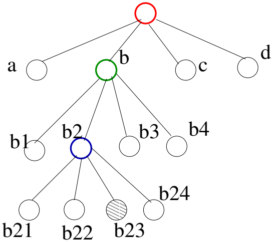

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- height H of the quadtree is $O(\lg n)$

  level of root is 0; for every $0 \le i \le H$, grid corresponding to level i is denoted with G^i and the length of any edge of G^i is $\frac{1}{2^i}$

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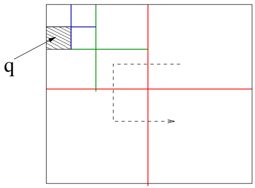

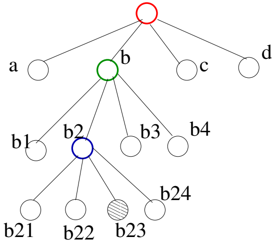

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- letting no node of the quadtree is empty, the number of nodes is $O(n \lg n)$

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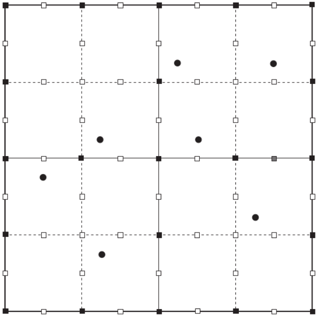
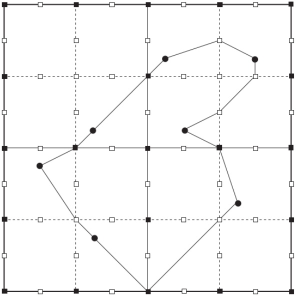
portals along the quadtree square sides



introduce equi-spaced $m = \frac{20H}{\epsilon}$ (which is $O(\frac{\lg n}{\epsilon})$) *portals* along each side of each quadtree square together with four at the corners of the same [3]

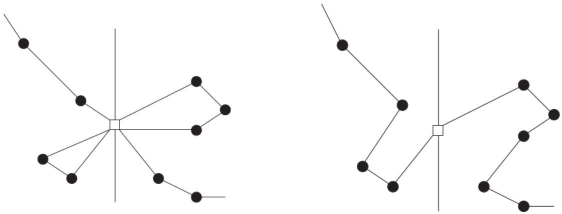
[3] choosing $m+1$ as a power of two, each portal on the sides of a level $i-1$ square are at the same location as a portal on the side of some level i square contained in the level $i-1$ square

requisite characteristics of a constrained tour



- tour is permitted to cross an edge e of a quadree square only via portals on e

requisite characteristics of a constrained tour (cont)



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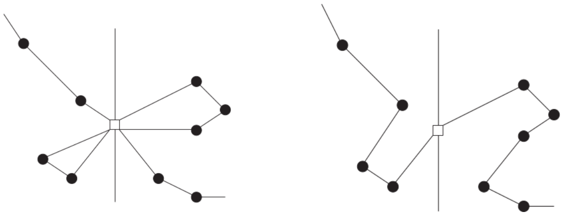

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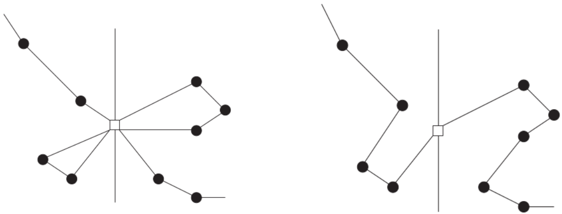

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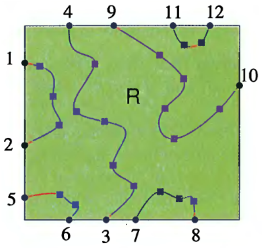
  is it possible to have such a tour while being a good apprx to an optimal tour?

nested parenthesis structure of portal-to-portal paths



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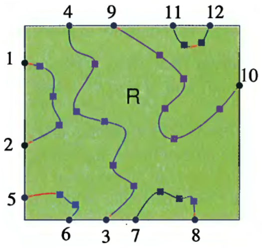

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- portal-to-portal paths follow parentization; hence, one can try all settings of parenthesis, translate these into possible layouts of paths and discard the ones that have intersecting paths

- therefore, the number of layouts of k portal-portal paths in grid square is the k^{th} Catalan number, which is $\frac{1}{k+1}\binom{2k}{k} = O(2^{2k})$

valid solutions of any grid square

- for every 0 to $4r$ portal combination of $4m+4$ portals on the boundary, considering that each chosen portal can be used 1 to 2 times, for every permutation p of the chosen portals, alternately mark portals in p with enter-exit $\rightarrow$ number of subproblems $\leq \sum_{i=0}^{8r} \binom{2(4m+4)}{i} i!$ i.e., $\left(\frac{1}{\epsilon} \lg n\right)^{O\left(\frac{1}{\epsilon}\right)}$

[4] a collection of subpaths (which intersect this square) of the apprx tour being constructed

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  - feasibility criteria: precisely satisfies the portal marking of that quadtree square; non-intersecting; subtours cover points belonging to quadtree square

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— to achieve efficiency, build the subtours in bottom-to-top fashion: using memoization of DP, from leaf nodes to root of the quadtree

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dynamic programming over squares of quadtree grid

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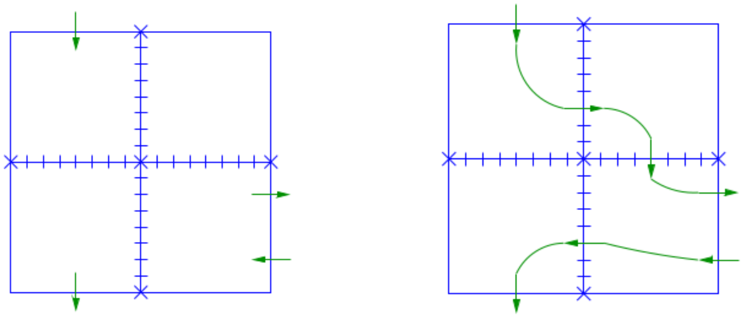
- base case: if a square has $O(\frac{1}{\epsilon})$ points of Q, then solve that subprobelm by brute force

DP: assemble subtours from children



- at every non-leaf node, for every chosen portal permutation, save these entries in the DP table: portal permutations of each child node which together caused the minimum cost subtour at the parent

time complexity

- number of subproblems to pursue at each quadtree node $\leq \sum_{i=0}^{8r} \binom{2(4m+4)}{i} i!$ i.e., $(\frac{1}{\epsilon} \lg n)^{O(\frac{1}{\epsilon})})$

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- subproblem at each quadtree node (ignoring the recursive subproblems) can be solved in $((\frac{1}{\epsilon} \lg n)^{O(\frac{1}{\epsilon})})^{O(1)}$ time
  - assembling: each subproblem solution of 1^{st} child with each ... with each subproblem solution of 4^{th} child
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- since there are $O(n \lg n)$ nodes in the quadtree, the time is $n(\frac{1}{\epsilon} \lg n)^{O(\frac{1}{\epsilon})}$ (including the time to brute-force enumeration at leaf nodes)

Outline

1. algorithm

2. analysis

factors causing errors

(i) building a tour π_{opt}^{Q} of points of Q instead of a tour π_{opt} of points of P

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We denote $\|\pi_{opt}\|$ with OPT and $\|\pi^Q_{opt}\|$ with OPT_Q.

error type (i): rounding points in P to get to Q

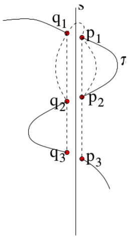
- The error in making π_{opt}^{Q} to touch points of P is

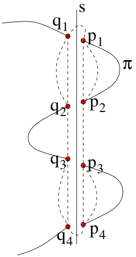
$$\leq n(2\frac{\sqrt{2}}{n\lceil\frac{32}{\epsilon}\rceil})$$

$$\leq \frac{4\epsilon}{32}$$

$$\leq \frac{\epsilon}{2}OPT \quad \text{(since } OPT \geq diam(P) \text{ which is } \tfrac{1}{4}\text{)}$$

patching lemma: patch the path with respect to a line segment





find an Eulerian tour in the graph

- replace a polyline π that crosses a line segment s at least three times with a polyline π' that crosses s at most twice: construct an Eulerian graph by introducing a few edges (see the above Fig.) to π
  $\|\pi'\| \leq \|\pi\| + (\sum_{i \text{ being odd}} \|p_i p_{i+1}\| + \|q_i q_{i+1}\|) + 2\|s\| \leq \|\pi\| + 4\|s\|$
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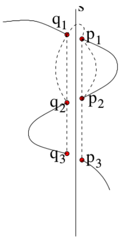

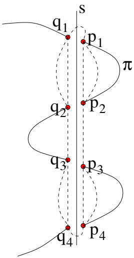

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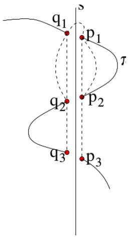

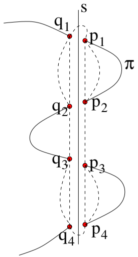

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- corollary: viewing the portal is of zero lengthed segment, any optimal solution need to use a portal at most twice (hence, we are allowing 2 intersections per portal in the apprx tour)

patching grid edges at every level of the quadtree

- while considering quadtree nodes in bottom-to-top[5], at every level $H \geq i \geq 1$[6], patch the tour π_{opt} w.r.t. every grid edge of G^i:

  if the current tour intersects e more than r times so that it intersects e at most twice after patching

[5] when we fix an edge of a grid so that the tour does not intersect it too many times, the number of times the patched tour crosses boundaries of higher-level nodes of the quadtree also goes down; similarly, the total number of crossings (of the tour with the grids) drop exponentially as we use larger and larger grids; thus requiring fewer fix-ups; thus, intuitively, one can think about all the patching happening in the bottom level of the quadtree

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  let π_i denote the resulting path after patching at G^i

  let π_{i+1} denote the path just before patching at G^i

  π_{H+1} is π_{opt}, and π_1 is the patched tour after patching at every level from H to 1

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exploiting sliding grid

randomly choose the unit grid origin from $[0, 0] \times [\frac{1}{2}, \frac{1}{2}]$:

- for every level $i \geq 1$ of the quadtree, the expected number of intersections of a polyline w with the vertical and horizontal edges of G^i is at most $2^{i+1}\|w\|$ $\leftarrow$ probabilistic argument is due to sliding grid

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- if a point p belongs to open edges of G^i then the point p belongs to open edges of G^{i-1} with probability $\frac{1}{2}$ (as every grid edge of G^i has probability $\frac{1}{2}$ to survive and be a grid edge of G^{i-1})

  ———————— (2)

error type (ii): patching π_{opt} at every level from bottom-to-top

F_i: number of patching operations performed at i^{th} level; Y_i: number of times the current tour crosses the edges of G^i just before considering G^i for patching; n_i: number of times the current tour crosses the edges of G^i after patching

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error type (iii): expected error due to snapping tour to portals

- as noted, after applying patching lemma to grid squares at all levels from bottom-to-top, the tour intersects with any edge of any grid square at most twice

[7] for $i \geq 1$, tour π_i (esp. π_1) has at least as many crossings with G_i as π_{opt} has with G_i

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- expected increase in length due to snapping π_1 to the portals of the quadtree (summed over all the levels)[8] $\leq E[\sum_{i=1}^{H} \frac{Z_i}{2^i(m+1)}] \leq \frac{2H}{m+1} OPT$

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overall apprx factor

- $(1+\frac{\epsilon}{2})(1+\frac{8}{r-2})(1+\frac{2H}{m+1})OPT$

  $\leq (1+\frac{\epsilon}{2})(1+\frac{\epsilon}{10})^2 OPT$ (by choosing $r=\frac{90}{\epsilon}=O(\frac{1}{\epsilon})$ and $m \geq \frac{20H}{\epsilon}=O(\frac{\lg n}{\epsilon})$)

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hence, the apprx factor in expectation is $(1+\epsilon)$ and the time complexity is $n(\frac{\lg n}{\epsilon})^{O(1/\epsilon)}$ i.e., randomized PTAS

take-home lessons

- PTAS in expectation
- rounding input points
- Steiner points (portals) along grid edges
- dynamic programming
- locally patching a path
- Hochbaum-Mass shifting grid technique
- applicable in various contexts: minimum Steiner tree, k-TSP, k-MST, Euclidean min-cost perfect matching

hence, the Godel award for the result!

other important results

- $O(n^{d+1} polylog)$ time PTAS — Arora '97
- $O(n \lg n)$ time PTAS — Rao and Smith '98
- NP-hard to get $(1+\epsilon)$-apprx in $\mathbb{R}^{O(\lg n)}$, for some $\epsilon > 0$ — Trevisan '97

references

- S. Har-Peled. Geometric Approximation Algorithms. American Mathematical Society, 2011.
- D. P. Williamson and D. B. Shmoys. The Design of Approximation Algorithms. Cambridge University Press, 2011.
- S. Arora. Polynomial time approximation schemes for Euclidean TSP and other geometric problems. Journal of ACM, 1998.

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- S. Arora. Polynomial time approximation schemes for Euclidean TSP and other geometric problems. Journal of ACM, 1998.

Parallel/further significant work -

- J. S. B. Mitchell. Guillotine subdivisions approximate polygonal subdivisions: A simple new method for the geometric TSP, k-MST, and related problems. SIAM Journal on Computing, 1999.
- S. B. Rao and W. D. Smith. Approximating geometric graphs via ”spanners” and ”banyans”. Symposium on Theory of Computing, 1998.