Computing a Euclidean shortest in the plane using visibility graphs

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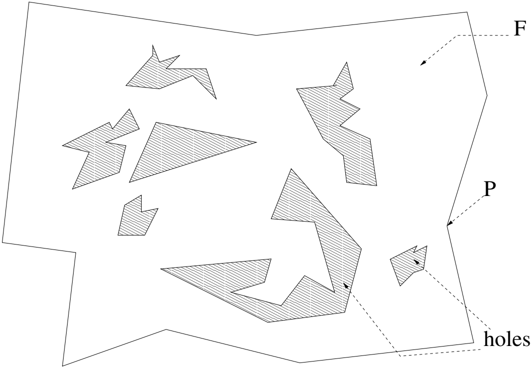
Outline

1. Problem description
2. Characterizations
3. Compute tangents that lie in F between convex holes
4. Apply a graph algorithm to find a SP
5. Conclusions

Polygonal domain

- A simple polygon P containing disjoint simple polygonal holes (a.k.a. obstacles) in R^2 is termed as the **polygonal domain** D.
- Polygon P sans interior of the holes is termed as the **free space** F.

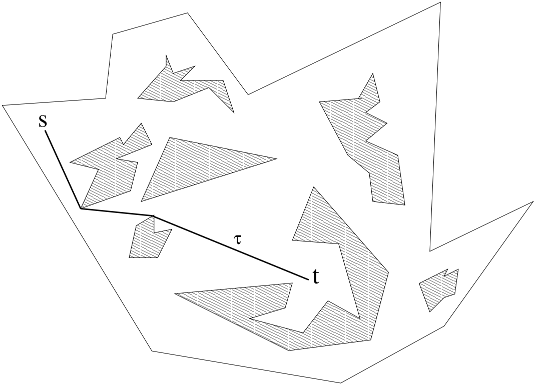


n: number of vertices

h: number of holes

Shortest paths in polygonal domains

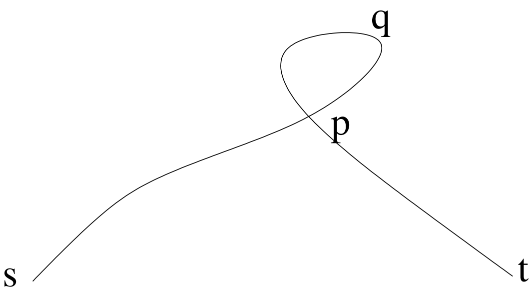
- Given D with two points $s, t \in F$, find *a* **Euclidean shortest path (SP)**, say SP_{st}, from s to t such that SP_{st} lies in F.



Applications in robot motion planning, route planning using GPS, VLSI wire routing etc.,

Shortest paths are simple paths

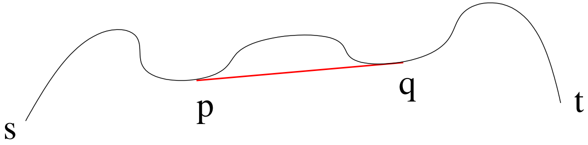
SP_{st} is a simple path.



- proof by contradiction

Optimal Substructure: SPs are locally shortest

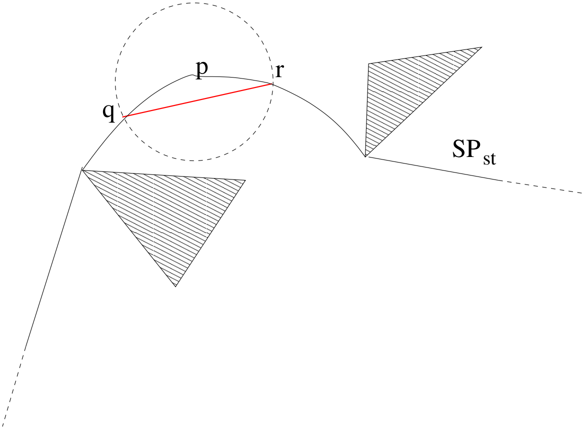
SP_{st} contains subpath L from p to $q \Rightarrow L$ must be SP_{pq}



- proof by contradiction

SP is polygonal

Every shortest path is a polygonal path.

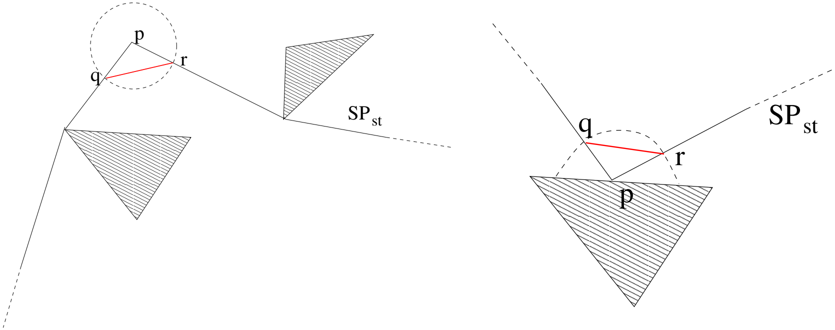


- proof by contradiction

Every internal vertex of SP is a vertex of D

No internal vertex of a SP can lie either:

- in the free space, or
- interior to an edge.



- proof by contradiction

Angle at every internal vertex is outward convex

proof by contradiction

Unique shortest paths in simple polygons

For two points s, t in a simple polygon, $SP(s, t)$ is unique.

- proof by contradiction

High Level Description

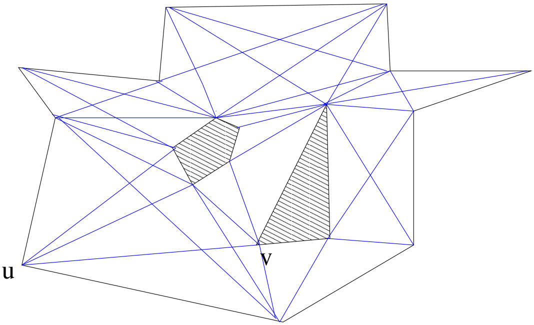
1. Compute a weighted graph G from D so that an edge $e \in SP_{st}$ in D then $e \in G$.
2. Compute SP_{st} in G.

Outline

Visibility Graph of D

The weighted undirected graph $VG_D(V, E')$ is defined over D such that:

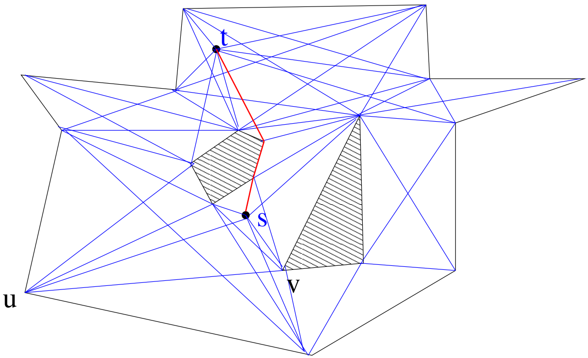
- V is the set of vertices in D,
- an edge $e(u, v) \in E'$ whenever u and v are visible to each other in D, and
- for every edge $e(u, v)$, the weight of e is the Euclidean distance along the line segment uv in D.



Geometric Shortest Paths and Visibility Graphs

Considering s and t as degenerate holes in D, $e \in SP_{st}$ in $D \Leftrightarrow e \in SP_{st}$ in VG_D.

- SP_{st} is polygonal with the internal vertices chosen from D, and
- Every edge $e(u, v)$ in SP_{st} belongs to F.

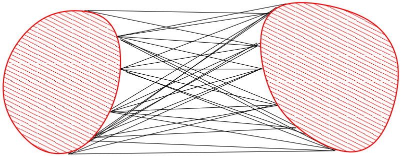


Time complexity is dominated by VG_D computation and $|E'|$.

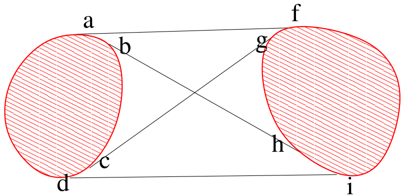
Tangent Visibility Graph

The **tangent visibility graph** $TVG_D(V, E)$ for D is defined whenever each hole in D is convex. It is same as $VG_D(V, E')$ except that:

> an edge $e(u, v) \in E$ iff uv is either an edge of a hole in D or *a tangent between two convex holes.*



$|E|$ is $\Omega(n^2)$.

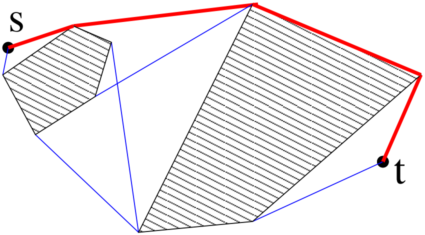


$|E|$ is $\Omega(h^2)$.

Computing tangents between two convex hulls CH' and CH'' takes $O(\lg |CH'| + \lg |CH''|)$ (from [Edelsbrunner, 1985]).

Geometric Shortest Paths and Tangent Visibility Graphs

Considering s and t as degenerate holes in D, $e \in SP_{st}$ in $D \Leftrightarrow e \in SP_{st}$ in TVG_D.



(Yet Another) High Level Description

Suppose that all holes are convex and the boundary of outer polygon is convex.

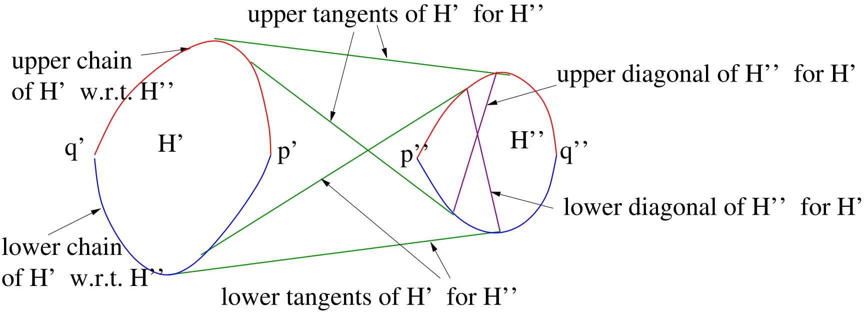
1. Compute the weighted tangent visibility graph *TVG* corresonding to *D*.
2. Find a shortest path from *s* to *t* in *TVG*.

Approach is due to [Rohnert, 1986].

Outline

Computing Tangents between every two Convex Holes

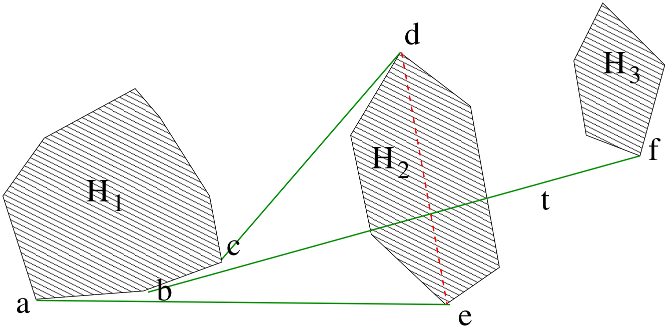
- Divide each convex chain into upper and lower convex chains.
- Compute all the four possible tangents between every two convex holes.



Time complexity: $O(h^2 \lg(n))$
from $\sum_{i,j} O(\lg |H'| + \lg |H''|)$

One lower tangent vs other lower tangents

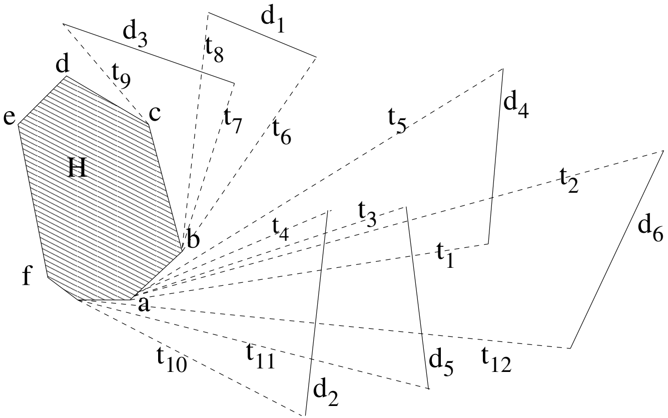
The lower tangent t of H_1 for H_3 intesects a hole H_2 iff t intersects the lower diagonal of H_2 for H_1.



t intersects H_2 iff t intersects de.

Computing lower tangents that lie in F

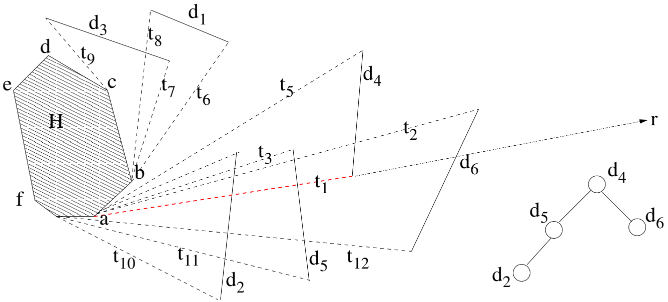
Traverse the boundary of H in counterclockwise order starting from any vertex a, and at each vertex v of H, add the lower tangents of H incident at v to the ordered list L according to their counterclockwise angle at v with the clockwise edge of H at vertex v.



$$L = \{t_1, t_2, \ldots, t_{12}\}$$

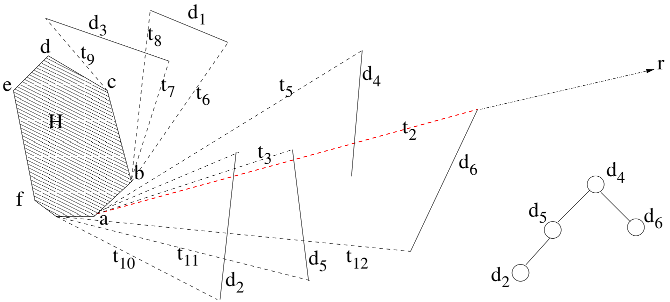
Computing lower tangents that lie in F (cont)

- Construct a BBST T to reflect the order in which the diagonals intersect the ray r.
- While exploring each such ray r, remove t_i from L if d corresponding to t_i is not the leftmost leaf of T.



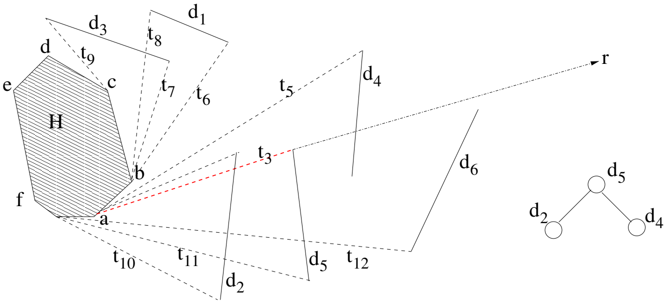
$$L = \{\mathbf{t_1}, t_2, \ldots, t_{12}\}$$

Computing lower tangents that lie in F (cont)



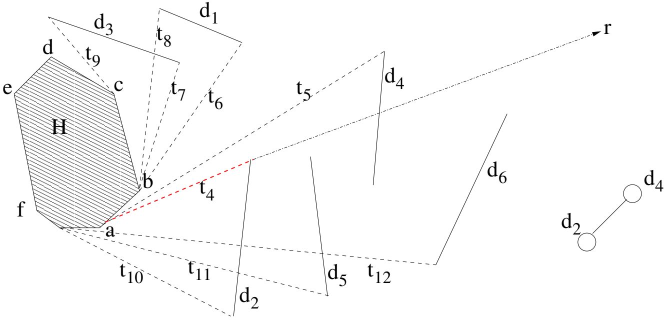
$$L = \{\mathbf{t_2}, \ldots, t_{12}\}$$

Computing lower tangents that lie in F (cont)



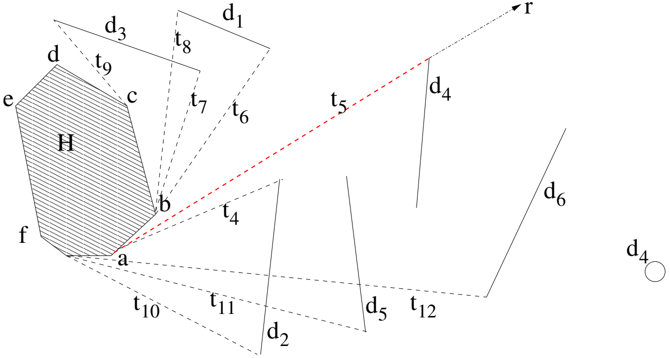
$$L = \{\mathbf{t_3}, \ldots, t_{12}\}$$

Computing lower tangents that lie in F (cont)



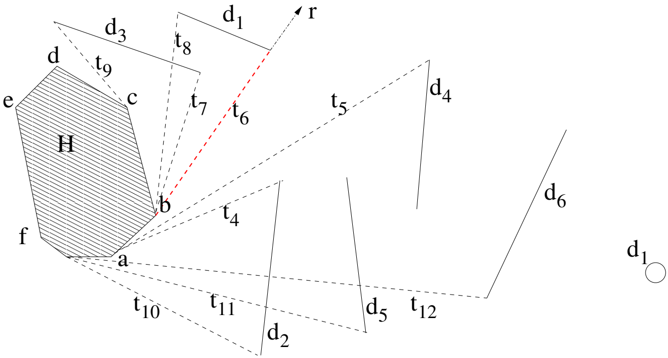
$$L = \{\mathbf{t_4}, \ldots, t_{12}\}$$

Computing lower tangents that lie in F (cont)



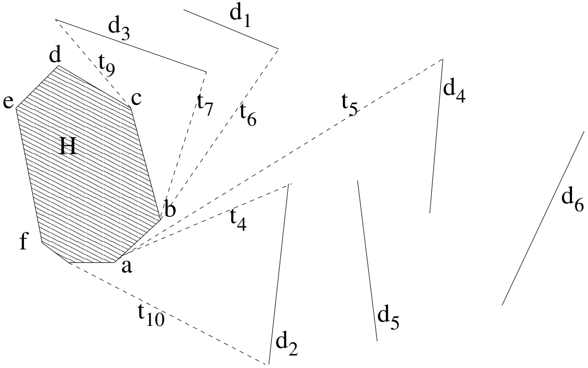
$$L = \{t_4, \mathbf{t_5}, \ldots, t_{12}\}$$

Computing lower tangents that lie in F (cont)



$$L = \{t_4, t_5, \mathbf{t_6}, \ldots, t_{12}\}$$

Computing lower tangents that lie in F (cont)



Output: $L = \{t_4, t_5, t_6, t_7, t_9, t_{10}\}$

- Time to process one hole: $O(|V_H| + h \lg h)$
- Time to process all holes: $O(n + h^2 \lg h)$

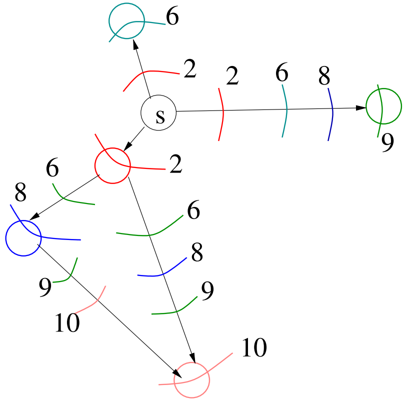
Computing upper tangents that lie in F (cont)

Applying similar procedure as in identifying lower tangents that lie in F:

- Time to process one hole: $O(|V_H| + h \lg h)$
- Time to process all holes: $O(n + h^2 \lg h)$

Outline

Computing SP in $TVG_D(V, E)$ using Dijkstra's Algorithm



- $|V|$ is n
- $|E|$ is $O(h^2 + n)$

- Applying Dijkstra's Algorithm to compute SP takes $O(h^2 + n + n \lg n)$.

Time complexity of the suggested algorithm

- Computing all possible tangents between every two holes: $O(h^2 \lg n)$
- Computing all tangents that lie in F: $O(n + h^2 \lg h)$
- Computing SP over tangent visibility graph: $O(h^2 + n \lg n)$

Total time: $O(n + (n + h^2) \lg n)$.

Assuming $h^2 > n$, it is $O(n + h^2 \lg n)$.

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Summary

- Building tangent visibility graph and running SP algorithm for graphs: $O(n + (n + h^2) \lg n)$ and $O(n)$ space.
- Running continous Dijkstra's algorithm in geometric domain: $O(T + h(\lg h)(\lg n))$ and $O(n)$ space.
- Problem 21 of The Open Problems Project (TOPP) of Computational Geometry which intends for a solution with $O(n + h \lg h)$ time and $O(n)$ space.

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