MA 102 (Multivariable Calculus)

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Date: April 19, 2013 Tutorial Sheet No. 7 R. Alam

Surface integrals, Stokes' Theorem, Divergence Theorem

(1) Let $R \subset \mathbb{R}^2$ be simply connected with positively oriented boundary ∂R and w be a C^2 scalar field on an open set containg R. Also let $\mathbf{n}$ be the unit outward normal field to the boundary ∂R. Then prove the following identities:

(a) $\iint_R \nabla^2 w(x,y)\, dA = \int_{\partial R} \frac{\partial w}{\partial \mathbf{n}}\, ds.$
(b) $\iint_R \left(w\nabla^2 w + \nabla w \cdot \nabla w\right)\, dA = \int_{\partial R} w\frac{\partial w}{\partial \mathbf{n}}\, ds.$
(c) $\int_{\partial R} \left(v\frac{\partial w}{\partial \mathbf{n}} - w\frac{\partial v}{\partial \mathbf{n}}\right)\, ds = \iint_R \left(v\nabla^2 w - w\nabla^2 v\right)\, dA.$

(2) Evaluate $\iiint_V f(x,y,z)dV$ for f and V specified below.
(a) $f(x,y,z) = \sqrt{x^2+z^2}$ and V is the solid region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$.
(b) $f(x,y,z) = z$ and V is the solid tetrahedron bounded by the coordinate planes and the plane $x + y + z = 1$. Also find the volume of V.
(c) $f(x,y,z) = \sqrt{x^2+y^2}$ and V is the solid region within the cylinder $x^2+y^2 = 1$ bounded above by $z = 4$ and below by $z = 1 - x^2 - y^2$.

(3) Consider the change of variable $x = u^2 - v^2, y = 2uv$ to evaluate the integral $\iint_D y dA$ where D is the region bounded by the parabolas $y^2 = 4 - 4x$ and $y^2 = 4 + 4x$, and the x-axis.

(4) Let $D \subset \mathbb{R}^2$ be simply connected with positively oriented boundary ∂D. Let F and G be C^1 vector fields on an open set containing D such that

$$\operatorname{curl} F = \operatorname{curl} G, \quad \operatorname{div} F = \operatorname{div} G \quad \text{on } D \cup \partial D$$

and $F \cdot \mathbf{n} = G \cdot \mathbf{n}$ on ∂D, where $\mathbf{n}$ is the unit outward normal to the curve ∂D. Show that $F = G$ on D.

(5) Let $C \subset \mathbb{R}^2$ be a positively oriented closed curve with unit outward normal field $\mathbf{n}$. Evaluate

$$\oint_C \nabla(x^2 - y^2) \cdot \mathbf{n}\, ds.$$

(6) Let $\Phi : \mathbb{R}^2 \to \mathbb{R}^3$ be a parametrization of a smooth surface S. Find the parametric equation of the tangent plane to S at $\Phi(u_0, v_0)$. In particular, consider $\Phi(u,v) := (u^2, v^2, u + 2v)$ and find the equation of tangent at $\Phi(u_0, v_0)$.

(7) Find a parametrization $\Phi(u,v)$ and the normal vector $\Phi_u \times \Phi_v$ for the following surfaces:
(a) The plane $x - y + 2z + 4 = 0$.
(b) The right circular cylinder $y^2 + z^2 = a^2$.
(c) The right circular cylinder of radius 1 whose axis is along the line $x = y = z$.

(8) Let $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ be C^1. Consider the surface $S = \text{Graph}(f)$. Show that the surface area of S is given

$$\text{Area}(S) = \iint_D \sqrt{1 + \|\nabla f\|^2}\, dA.$$

Find the surface area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.

(9) Let $g : D \subset \mathbb{R}^2 \to \mathbb{R}$ be C^1. Consider the surface $S = \text{Graph}(g)$ with outward normal field $\mathbf{n}$. Let $F = (P, Q, R)$ be a continuous vector field on D. Then show that

$$\iint_S F \bullet \mathbf{n} dS = \iint_D (R - Pg_x - Qg_y) dA.$$

(10) Compute the surface area of the portion of the paraboloid $x^2 + z^2 = 2ay$ which is between the planes $y = 0,\ y = a$.

(11) A fluid flow has flux density vector $F(x, y, z) = x\mathbf{i} + (2x + y)\mathbf{j} + z\mathbf{k}$. Let S denote the hemisphere $x^2 + y^2 + z^2 = 1,\ z \geq 0$, and $\mathbf{n}$ be the unit outward normal field on S. Then calculate the mass of the fluid flowing though S in unit time in the direction of $\mathbf{n}$.

(12) Let R be a region bounded by a piecewise smooth closed surface S with outward unit normal $\mathbf{n} = n_1\mathbf{i} + n_2\mathbf{j} + n_3\mathbf{k}$. Let $(u, v) : R \to \mathbb{R}$ be C^1 functions. Show that

$$\iiint_R u \frac{\partial v}{\partial x}\, dV = -\iiint_R v \frac{\partial u}{\partial x}\, dV + \iint_{\partial R} uvn_1\, dS.$$

(13) Let C be the curve obtained by intersecting the plane $y+z = 2$ with the cylinder $x^2+y^2 = 1$. (Orient the curve counterclockwise when viewed from the above.) Use Stokes' theorem to evaluate the line integral $\int_C F \cdot\ d\mathbf{r}$, where $F(x, y, z) = -y^2\mathbf{i} + x\mathbf{j} + z^2\mathbf{k}$.

(14) Use Stokes' Theorem to compute the integral $\iint_S \text{curl}\, F \cdot\ \mathbf{n} dS$, where $F(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ and S is the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies inside the cylinder $x^2 + y^2 = 1$ and above the xy-plane.

(15) Use Divergence theorem to evaluate $\iint_S F \bullet d\mathbf{S}$, where $F = (xy, y^2 + e^{xz^2}, \sin(xy))$ and S is the boundary of the solid region V bounded by the parabolic cylinder $z = 1 - x^2$ and the planes $z = 0, y = 0$ and $y + z = 2$.

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