MA 102 (Multivariable Calculus)

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Date: April 12, 2013 Tutorial Sheet No. 6 R. Alam

Line integrals, Double integrals, Green's Theorem

(1) Evaluate the line integral $\int_\Gamma F \bullet \mathbf{dr}$ of the vector field F given below.

(a) $F(x,y) := (x^2+2xy,\ y^2-2xy)$ from $(-1,1)$ to $(1,1)$ along $y=x^2$.

(b) $F(x,y) := (x^2-y^2,\ x-y)$ and $\Gamma : \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in the counterclockwise direction.

(2) Evaluate the line integral $\displaystyle\int_\Gamma \frac{(x+y)dx-(x-y)dy}{x^2+y^2}$ along $\Gamma : x^2+y^2=a^2$ traversed once in the counter clockwise direction.

(3) Evaluate the line integral $\displaystyle\int_\Gamma (ydx+zdy+xdz)$, where Γ is the intersection of two surfaces $z=xy$ and $x^2+y^2=1$ traversed once in a direction that appears counter clockwise when viewed from high above the xy-plane.

(4) Consider the helix $\mathbf{r}(t)=(a\cos t,\ a\sin t, ct)$, where $a>0$. Parametrize the helix in terms of arc length.

(5) Evaluate the line integral $\displaystyle\int_\Gamma \frac{x^2ydx-x^3dy}{(x^2+y^2)^2}$, where Γ is the square with vertices $(\pm1,\pm1)$ oriented in the counterclockwise direction.

(6) Evaluate the line integral $\displaystyle\int_\Gamma \frac{dx+dy}{|x|+|y|}$, where Γ is the square with vertices $(1,0),(0,1)$, $(-1,0)$ and $(0,-1)$ oriented in the counterclockwise direction.

(7) The force field $F=(xy,x^6y^2)$ moves a particle from $(0,0)$ to the line $x=1$ along $y=ax^b$ where $a,b>0$. If the work done is independent of b, find the value of a.

(8) Determine whether or not the vector field $F(x,y)=(3xy,x^3y)$ is a gradient field (conservative) on any open subset of $\mathbb{R}^2$.

(9) Determine whether the differential form $e^{xy}dx+e^{x+y}dy$ is exact in $\mathbb{R}^2$. If so, find a scalar potential f such that $df=e^{xy}dx+e^{x+y}dy$, that is, $F(x,y)=(e^{xy},e^{x+y})=\nabla f$.

(10) Determine which of the following vector fields F in $\mathbb{R}^2$ is conservative and find a scalar potential when it exists.
(a) $F(x,y)=(\cos(xy)-xy\sin(xy),x^2\sin(xy))$.
(b) $F(x,y)=(xy,xy)$.
(c) $F(x,y)=(x^2+y^2,\ 2xy)$.

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(11) Let $S = \mathbb{R}^2 \backslash \{(0,0)\}$. Let $F(x,y) = (\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}) =: (P(x,y), Q(x,y))$. Show that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ on S while F is not a gradient of a scalar field on S.

(12) Evaluate the double integral $\iint_R f(x,y)\, dxdy$ for f and R given below.
  (a) $f(x,y) := (1-x)$ and R is square $[0,1] \times [0,1]$.
  (b) $f(x,y) := x^2 + y^2$ and $R = [-1,1] \times [0,1]$.
  (c) $f(x,y) := x^2 + y$ and R is the square $[0,1] \times [0,1]$.
  (d) $f(x,y) := \sin(x+y)$ and R is the square $[0,\pi] \times [0,\pi]$.

(13) Find the volume of the solid enclosed between the graph of $f(x,y) = x^2+y^2$ and the planes $x = 0, x = 3, y = -1, y = 1$.

(14) Verify Green's theorem in each of the following cases:
  (a) $f(x,y) := -xy^2$; $g(x,y) := x^2y$; the region R is given by $x \geq 0, 0 \leq y \leq 1 - x^2$.
  (b) $f(x,y) := 2xy$; $g(x,y) := e^x + x^2$; the region R is the triangle with vertices $(0,0), (1,0)$ and $(1,1)$.

(15) Evaluate $\int_\Gamma (y^2dx + xdy)$ using Green's theorem, where Γ is boundary of R and
  (a) R is the square with vertices $(0,0), (0,2), (2,2), (2,0)$.
  (b) R is the square with vertices $(\pm 1, \pm 1)$.
  (c) R is the disc of radius 2 and center $(0,0)$. (Specify the orientation of your curve.)

(16) Use Green's theorem to show that the area enclosed by a simple closed curve Γ in polar coordinates is given by

$$A = \frac{1}{2}\int_C r^2 d\theta.$$

Use this to compute the area enclosed by the following curves:
  (i) The cardioid: $r = a(1-\cos\theta), \quad 0 \leq \theta \leq 2\pi$.
  (ii) The lemniscate $r^2 = a^2 \cos 2\theta, \quad -\pi/4 \leq \theta \leq \pi/4$.

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