MA 102 (Multivariable Calculus)

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Date: April 05, 2013 Tutorial Sheet No. 5 R. Alam

Implicit derivative, constrained extrema, vector fields, arclength

(1) Consider the equation $e^{2x-y}+\cos(x^2+xy)-2-2y=0$ for $(x,y)\in\mathbb{R}^2$. Can the solutions be written as $y=\phi(x)$ and $x=\psi(y)$ in a neighbourhood of 0? If so, compute the derivatives $\phi'(0)$ and $\psi'(0)$.

(2) Show that around the point $(0,1,1)$, the equation $xy - z\log y + e^{xz} = 1$ can be solved locally as $y=f(x,z)$ but cannot be solved locally as $z=g(x,y)$. Find $f_x(0,1)$ and $f_z(0,1)$.

(3) Let S be a surface given by $x^3+3y^2+8xz^2-3z^3y-1=0$. Find all points $(x_0,y_0,z_0)\in\mathbb{R}^3$ such that S is represented as a graph of a differentiable function $z=f(x,y)$ in a neighbourhood of (x_0,y_0,z_0).

(4) Let $f,g:\mathbb{R}^n\to\mathbb{R}$ be C^1 scalar fields. Show that (a) $\nabla(fg)=f\nabla g+g\nabla f$, (b) $\nabla f^m = mf^{m-1}\nabla f$ and (c) $\nabla(f/g)=(g\nabla f - f\nabla g)/g^2$ whenever $g\neq 0$.

(5) Let F and G be vector fields in $\mathbb{R}^3$ and $f:\mathbb{R}^3\to\mathbb{R}$ be a C^1 scalar field. Then show that:

(a) $\operatorname{div}(F+G)=\operatorname{div}F+\operatorname{div}G$ and $\operatorname{curl}(F+G)=\operatorname{curl}F+\operatorname{curl}G$,
(b) $\operatorname{div}(fG)=f\operatorname{div}G+G\bullet\nabla f$ and $\operatorname{curl}(fG)=f\operatorname{curl}G+\nabla f\times G$,
(c) $\operatorname{div}(F\times G)=G\bullet\operatorname{curl}F-F\bullet\operatorname{curl}G$ and $\operatorname{curl}\operatorname{curl}F=\nabla\operatorname{div}F-\nabla^2F$.

(6) Let $\mathbf{r}=(x,y,z)$ and $r=\|\mathbf{r}\|=\sqrt{x^2+y^2+z^2}$. Then show that

(a) $\nabla r=\dfrac{\mathbf{r}}{r}$ and $\nabla(\dfrac{1}{r})=\dfrac{-\mathbf{r}}{r^3}$ for $r\neq 0$.
(b) $\operatorname{div}(r^m\mathbf{r})=(m+3)r^m$
(c) $\operatorname{curl}(r^m\mathbf{r})=0$ and $\operatorname{div}\left(\nabla\dfrac{1}{r}\right)=0$ for $r\neq 0$.

(7) Find the extrema of the function $f(x,y)=x^2+2y^2$ on the disk $x^2+y^2\leq 1$.

(8) Find a point on the ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$ that is closest to $(0,0,0)$.

(9) Find the maximum and minimum of $f(x,y)=5x-3y$ subject to the constraint

$$x^2+y^2=136.$$

(10) Find the global maximum (also called absolute maximum) of $f(x,y):=xy$ on the unit disk $x^2+y^2\leq 1$.

(11) Assume that among all rectangular boxes with fixed surface area of 10 square meters there is a box of largest possible volume. Find the dimensions of the optimum box.

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(12) Find the arclength of parabolic arc $\gamma(t) := (t, t^2)$ for $t \in [0, 4]$.

(13) Find the velocity, the speed and the arclength of the cycloid $\gamma(t) := (t - \sin t,\, 1 - \cos t)$ for $t \in [0, 2\pi]$.

(14) A billiard ball on a square table follows the path $\gamma : [-1, 1] \to \mathbb{R}^3$ given by $\gamma(t) := (|t|,\, |t - 1/2|,\, 0)$. Find the distance travelled by the ball.

(15) Find the arclength of the path $\gamma(t) := (t,\, t \sin t, t \cos t)$ between $(0, 0, 0)$ and $(\pi, 0, -\pi)$.

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