

MA 102 (Multivariable Calculus)

IIT Guwahati

Date: March 22, 2013

Tutorial Sheet No. 4

R. Alam

Chain rule, tangent and normal, Jacobian matrix

- (1) Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be such that $f(tx) = t^m f(x)$ for all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$, where m is a nonnegative integer. If f is differentiable then show that $\langle x, \nabla f(x) \rangle = m f(x)$.
- (2) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be continuous. Define $F, G : \mathbb{R}^2 \rightarrow \mathbb{R}$ by $F(x, y) := \int_0^{x+y} f(t) dt$ and $G(x, y) := \int_0^{xy} f(t) dt$. Show that F and G are differentiable and determine $DF(x, y)$ and $DG(x, y)$.
- (3) Let $f(x, y, z) = x^2 + 2xy - y^2 + z^2$. Find the gradient of f at $(1, -1, 3)$ and the equations of the tangent plane and the normal line to the surface $f(x, y, z) = 7$ at $(1, -1, 3)$.
- (4) Find $D_u f(2, 2, 1)$, where $f(x, y, z) = 3x - 5y + 2z$ and u is the unit vector in the direction of outward normal to the sphere $x^2 + y^2 + z^2 = 9$ at $(2, 2, 1)$.
- (5) Find the equation of the tangent plane to the graphs of the following functions at the given point:
(a) $f(x, y) := x^2 - y^4 + e^{xy}$ at the point $(1, 0, 2)$
(b) $f(x, y) = \tan^{-1} \frac{y}{x}$ at the point $(1, \sqrt{3}, \frac{\pi}{3})$.
- (6) Check the following functions for differentiability and the derivative matrix (Jacobian Matrix).
(a) $f(x, y) = (e^{x+y} + y, xy^2)$ (b) $f(x, y) = (x^2 + \cos y, e^x y)$ (c) $f(x, y, z) = (ze^x, -ye^z)$.
- (7) Let $z = x^2 + y^2$, and $x = 1/t, y = t^2$. Compute $\frac{dz}{dt}$ by (a) expressing z explicitly in terms of t and (b) chain rule.
- (8) Let $w = 4x + y^2 + z^3$ and $x = e^{rs^2}, y = \log \frac{r+s}{t}, z = rst^2$. Find $\frac{\partial w}{\partial s}$.
- (9) Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be given by $f(0, 0) := 0$ and, for $(x, y) \neq (0, 0)$, $f(x, y) := xy \frac{x^2 - y^2}{x^2 + y^2}$.
(a) Show that $\frac{\partial f}{\partial y}(x, 0) = x$ for $x \in \mathbb{R}$ and $\frac{\partial f}{\partial x}(0, y) = -y$ for $y \in \mathbb{R}$.
(b) Show that $\frac{\partial^2 f}{\partial x \partial y}(0, 0) \neq \frac{\partial^2 f}{\partial y \partial x}(0, 0)$.
- (10) Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be twice continuously differentiable. Show that
$$\lim_{h \rightarrow 0} \frac{f(x+h) - [f(x) + \langle \nabla f(x), h \rangle + \frac{1}{2} \langle H_f(x)h, h \rangle]}{\|h\|^2} = 0,$$
where $H_f(x)$ is the Hessian of f at x .
- (11) Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be twice continuously differentiable and $x = r \cos \theta, y = r \sin \theta$. Show that
$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial f}{\partial r} + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2}.$$

**** End ****