Lecture 4:
Partial and Directional derivatives, Differentiability

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Differential Calculus

**Task:** Extend differential calculus to the functions:

- **Case I:** $f : A \subset \mathbb{R}^n \rightarrow \mathbb{R}$
- **Case II:** $f : A \subset \mathbb{R} \rightarrow \mathbb{R}^n$
- **Case III:** $f : A \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$

**Question:** What does it mean to say that $f$ is differentiable?
Parametric curve $\mathbf{r} : \mathbb{R} \to \mathbb{R}^n$

A continuous function $\mathbf{r} : [a, b] \subset \mathbb{R} \to \mathbb{R}^n$ is called a **parametric curve** in $\mathbb{R}^n$. The curve $\Gamma := \mathbf{r}([a, b])$ is parameterized by $\mathbf{r}(t)$.

Examples:

- $\mathbf{r} : \mathbb{R} \to \mathbb{R}^n$ given by $\mathbf{r}(t) := a + t\mathbf{b}$ parameterizes a line in $\mathbb{R}^n$ passing through $a$ in the direction of $\mathbf{b}$.

- $\mathbf{r} : [0, 2\pi] \to \mathbb{R}^3$ given by $\mathbf{r}(t) := (\cos t, \sin t, t)$ parameterizes a circular helix.

- $\mathbf{r} : [0, 2\pi] \to \mathbb{R}^2$ given by $\mathbf{r}(t) := (\cos t, \sin t)$ parameterizes the circle $x^2 + y^2 = 1$. 
Figure: Line $\mathbf{r}(t) = \mathbf{p}_0 + t\mathbf{v}$
Figure: Helix \( r(t) = (4 \cos t, 4 \sin t, t) \)
Figure: Plane curve $r(t) = (t - 2\sin t, t^2)$
Figure: Ellipse $\mathbf{r}(t) = (6 \cos t, 3 \sin t)$
Differentiability of \( \mathbf{r} : \mathbb{R} \to \mathbb{R}^n \)

**Definition:** Let \( \mathbf{r} : (a, b) \subset \mathbb{R} \to \mathbb{R}^n \) and \( t_0 \in (a, b) \). If

\[
\mathbf{r}'(t_0) = \frac{d\mathbf{r}}{dt}(t_0) := \lim_{t \to t_0} \frac{\mathbf{r}(t) - \mathbf{r}(t_0)}{t - t_0}
\]

exists then \( \mathbf{r} \) is differentiable at \( t_0 \). The derivative \( \mathbf{r}'(t_0) \) is called the velocity vector.

**Fact:**

- \( \mathbf{r}(t) = (r_1(t), \ldots, r_n(t)) \), where \( r_i : (a, b) \to \mathbb{R} \).

- \( \mathbf{r} \) is differentiable at \( t_0 \) \iff each \( r_i \) is differentiable at \( t_0 \), \( i = 1, 2, \ldots, n \). Further, \( \mathbf{r}'(t_0) = (r'_1(t_0), \ldots, r'_n(t_0)) \).

- \( \mathbf{r} \) differentiable at \( t_0 \Rightarrow \mathbf{r} \) continuous at \( t_0 \).
Sum and product rules

**Fact:** Let $f, g : (a, b) \subset \mathbb{R} \to \mathbb{R}^n$ be differentiable at $t_0 \in (a, b)$. Then for $\alpha \in \mathbb{R}$

1. $f + g$ and $\alpha f$ are differentiable at $t_0$. Further, $(f + g)'(t) = f'(t_0) + g'(t_0)$ and $(\alpha f)'(t_0) = \alpha f'(t_0)$.

2. $f \circ g$ defined by $(f \circ g)(t) := \langle f(t), g(t) \rangle$ is differentiable at $t_0$ and

   $$(f \circ g)'(t_0) = f'(t_0) \circ g(t_0) + f(t_0) \circ g'(t_0).$$
Velocity and tangent vectors

Let \( \mathbf{r} : (a, b) \to \mathbb{R}^n \) be differentiable. Then treating \( \mathbf{r}(t) \) as the position of a moving object at time \( t \), we have

\[
\text{scaled secant} = \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} \to \mathbf{r}'(t) \quad \text{as} \quad \Delta t \to 0.
\]

But scaled secant \( \to \) tangent vector to the curve at \( \mathbf{r}(t) \) as \( \Delta t \to 0 \).

Thus velocity vector \( \mathbf{v}(t) := \mathbf{r}'(t) \) is tangent to the curve at \( \mathbf{r}(t) \).

If \( \mathbf{r}(t) := (\cos t, \sin t) \) then \( \mathbf{v}(t) = \mathbf{r}'(t) = (-\sin t, \cos t) \).
Partial derivatives of $f : \mathbb{R}^2 \to \mathbb{R}$

Let $f : \mathbb{R}^2 \to \mathbb{R}$ and $(a, b) \in \mathbb{R}^2$. Then

$$\frac{\partial f}{\partial x}(a, b) := \lim_{t \to 0} \frac{f(a + t, b) - f(a, b)}{t},$$

when exists, is called partial derivative of $f$ at $(a, b)$ w.r.t to the first variable.

Other notations for $\frac{\partial f}{\partial x}(a, b)$ :

$f_x(a, b), \partial_x f(a, b), \partial_1 f(a, b)$.

Partial derivative $\frac{\partial f}{\partial y}(a, b)$ w.r.t. the second variable is defined similarly.
Partial derivatives of $f : \mathbb{R}^n \to \mathbb{R}$

Let $f : \mathbb{R}^n \to \mathbb{R}$ and $a \in \mathbb{R}^n$. Then

$$
\frac{\partial f}{\partial x_i}(a) := \lim_{t \to 0} \frac{f(a + te_i) - f(a)}{t},
$$

when exists, is called partial derivative of $f$ at $a$ w.r.t to the $i$-th variable.

Other notations for $\frac{\partial f}{\partial x_i}(a)$:

$f_{x_i}(a), \partial_{x_i}f(a), \partial_i f(a)$.

If $\partial_i f(a)$ exists for $i = 1, 2, \ldots, n$, then $f$ is said to have first order partial derivatives at $a$. 
Examples

- Consider $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(0,0) := 0$ and $f(x,y) := xy/(x^2 + y^2)$ for $(x,y) \neq (0,0)$. Then

$$\partial_1 f(0,0) = \partial_2 f(0,0) = 0$$

even though $f$ is NOT continuous at $(0,0)$.

- Consider $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(0,0) = 0$ and

$$f(x,y) := \begin{cases} 
  x \sin(1/y) + y \sin(1/x) & \text{if } x \neq 0, y \neq 0, \\
  x \sin(1/x) & \text{if } x \neq 0, y = 0, \\
  y \sin(1/y) & \text{if } x = 0, y \neq 0.
\end{cases}$$

Then $f$ is continuous at $(0,0)$ but neither $\partial_1 f(0,0)$ nor $\partial_2 f(0,0)$ exists.

Moral: Partial derivatives $\not\Rightarrow$ continuity $\not\Rightarrow$ Partial derivatives
Let \( f, g : \mathbb{R}^n \to \mathbb{R} \) and \( a \in \mathbb{R}^n \). Suppose \( \partial_i f(a) \) and \( \partial_i g(a) \) exist. Then

- \( \partial_i(\alpha f)(a) = \alpha \partial_i f(a) \) for \( \alpha \in \mathbb{R} \),
- \( \partial_i(f + g)(a) = \partial_i f(a) + \partial_i g(a) \),
- \( \partial_i(fg)(a) = \partial_i f(a)g(a) + f(a)\partial_i g(a) \).
- If \( h : \mathbb{R} \to \mathbb{R} \) is differentiable at \( f(a) \) then \( \partial_i(h \circ f)(a) \) exists and \( \partial_i(h \circ f)(a) = h'(f(a))\partial_i f(a) \).
Gradient of $f : \mathbb{R}^n \to \mathbb{R}$

Define $\phi_i : \mathbb{R} \to \mathbb{R}$ by $\phi_i(t) := f(a + t e_i)$. Then

$$\partial_i f(a) = \lim_{t \to 0} \frac{\phi_i(t) - \phi_i(0)}{t} = \phi'_i(0) = \frac{d}{dt} f(a + t e_i) \big|_{t=0},$$

rate of change of $f$ at $a$ in the direction $e_i$.

Suppose partial derivatives of $f : \mathbb{R}^n \to \mathbb{R}$ exist at $a \in \mathbb{R}^n$. Then the vector

$$\nabla f(a) := (\partial_1 f(a), \ldots, \partial_n f(a)) \in \mathbb{R}^n$$

is called the **gradient** of $f$ at $a$. 
Figure: Graph of $z = f(x, y)$ and geometric interpretation of $\partial_x f(x_0, y_0)$. 
Figure: Graph of $z = f(x, y)$ and geometric interpretation of $\partial_y f(x_0, y_0)$. 
Directional derivatives of $f : \mathbb{R}^n \to \mathbb{R}$

Let $f : \mathbb{R}^n \to \mathbb{R}$ and $a \in \mathbb{R}^n$. Also let $u \in \mathbb{R}^n$ with $\|u\| = 1$. Then the limit, when exists,

$$D_uf(a) := \lim_{t \to 0} \frac{f(a + tu) - f(a)}{t} = \frac{d}{dt} f(a + tu) |_{t=0},$$

is called directional derivative of $f$ at $a$ in the direction $u$.

- $D_uf(a)$, also denoted by $\frac{\partial f}{\partial u}(a)$, is the rate of change of $f$ at $a$ in the direction $u$.
Properties of directional derivatives

Let \( f : \mathbb{R}^n \to \mathbb{R} \) and \( a \in \mathbb{R}^n \). Also let \( u \in \mathbb{R}^n \) with \( \|u\| = 1 \).

Then

- Sum, product and chain rule similar to those of \( \partial_i f(a) \) hold for \( D_u f(a) \).

- If \( D_u f(a) \) exists for all nonzero \( u \in \mathbb{R}^n \) then \( f \) is said to have directional derivatives in all directions.

- Obviously \( \partial_i f(a) = D_{e_i} f(a) \). Hence \( D_u f(a) \) exists in all directions \( u \Rightarrow \partial_i f(a) \) exist for \( i = 1, 2, \ldots, n \).
Examples

1. Consider $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ given by $f(x, y) := \sqrt{|xy|}$. Then
   $\partial_1 f(0, 0) = 0 = \partial_2 f(0, 0)$ and $f$ is continuous at $(0, 0)$. However, $D_u f(0, 0)$ does NOT exist for $u_1 u_2 \neq 0$.

2. Consider $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ given by $f(0, 0) = 0$ and
   
   $f(x, y) := \frac{x^2 y}{x^4 + y^2}$ if $(x, y) \neq (0, 0)$. Then $f$ is NOT continuous at $(0, 0)$, $\partial_1 f(0, 0) = 0 = \partial_2 f(0, 0)$ and $D_u f(0, 0)$ exits for all $u$. Further, $D_u f(0, 0) = u_1^2 / u_2$ for $u_1 u_2 \neq 0$.

**Moral:** Partial derivatives $\not\Rightarrow$ Directional derivative $\not\Rightarrow$ Continuity $\not\Rightarrow$ Directional derivative.