Singularities
Behavior of following functions $f$ at 0:

- $f(z) = \frac{1}{z^9}$
- $f(z) = \frac{\sin z}{z}$
- $f(z) = \frac{e^z - 1}{z}$
- $f(z) = \frac{1}{\sin \left(\frac{1}{z}\right)}$
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In the above we observe that all the functions are not analytic at 0, however in every neighborhood of 0 there is a point at which $f$ is analytic.
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Definition: The point \( z_0 \) is called a **singular point** or **singularity of \( f \)** if \( f \) is not analytic at \( z_0 \) but every neighborhood of \( z_0 \) contains at least one point at which \( f \) is analytic.

- Here \( \frac{e^z - 1}{z}, \frac{1}{z^2}, \sin \frac{1}{z}, \log z \) etc. has singularity at \( z = 0 \).
- \( \bar{z}, |z|^2, \text{Re} \, z, \text{Im} \, z, z \text{Re} \, z \) are nowhere analytic. That does not mean that every point of \( \mathbb{C} \) is a singularity.

A singularities are classified into **TWO** types:

1. A singular point \( z_0 \) is said to be an **isolated singularity or isolated singular point** of \( f \) if \( f \) is analytic in \( B(z_0, r) \setminus \{z_0\} \) for some \( r > 0 \).

2. A singular point \( z_0 \) is said to be an **non-isolated singularity** if \( z_0 \) is not an isolated singular point.

- \( \frac{\sin z}{z}, \frac{1}{z^2}, \sin(\frac{1}{z}) \) (0 is isolated singular point).
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If $f$ has an isolated singularity at $z_0$, then $f$ is analytic in $B(z_0, r) \setminus \{z_0\}$ for some $r > 0$. In this case $f$ has the following Laurent series expansion:

$$f(z) = \cdots \frac{a_{-n}}{(z - z_0)^n} + \cdots + \frac{a_{-1}}{(z - z_0)} + a_0 + a_1(z - a) + a_2(z - z_0)^2 + \cdots.$$ 

- If all $a_{-n} = 0$ for all $n \in \mathbb{N}$, then the point $z = z_0$ is a removal singularity.
- The point $z = z_0$ is called a pole if all but a finite number of $a_{-n}$'s are non-zero. If $m$ is the highest integer such that $a_{-m} \neq 0$, then $z_0$ is a Pole of order $m$.
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Lecture 16

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Removable singularities

The following statements are equivalent:

1. $f$ has a removable singularity at $z_0$.
2. If all $a_{-n} = 0$ for all $n \in \mathbb{N}$.
3. $\lim_{z \to z_0} f(z)$ exists and finite.
4. $\lim_{z \to z_0} (z - z_0)f(z) = 0$.
5. $f$ is bounded in a deleted neighborhood of $z_0$.

The function $\frac{\sin z}{z}$ has removable singularity at $0$. 
The following statements are equivalent:

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The function \( \frac{\sin z}{z} \) has removable singularity at 0.
The following statements are equivalent:

- \( f \) has a pole of order \( m \) at \( z_0 \).

- \( f(z) = \frac{g(z)}{(z - z_0)^m} \), \( g \) is analytic at \( z_0 \) and \( g(z_0) \neq 0 \).

- \( \frac{1}{f} \) has a zero of order \( m \).

- \( \lim_{z \to z_0} |f(z)| = \infty \).

- \( \lim_{z \to z_0} (z - z_0)^{m+1} f(z) = 0 \)

- \( \lim_{z \to z_0} (z - z_0)^m f(z) \) has removal singularity at \( z_0 \).
The following statements are equivalent:

- $f$ has a essential singularity at $z_0$.
- The point $z_0$ is neither a pole nor removable singularity.
- $\lim_{{z \to z_0}} f(z)$ does not exists.
- Infinitely many terms in the principal part of Laurent series expansion around the point $z_0$.

Limit point of zeros is isolated essential singularity. For example:

$$f(z) = \sin \frac{1}{z}$$
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$$f(z) = \sin \frac{1}{z}$$
Let $f$ be a complex valued function. Define another function $g$ by

$$g(z) = f\left(\frac{1}{z}\right).$$

Then the nature of singularity of $f$ at $z = \infty$ is defined to be the the nature of singularity of $g$ at $z = 0$.

- $f(z) = z^3$ has a pole of order 3 at $\infty$.
- $e^z$ has an essential singularity at $\infty$.
- An entire function $f$ has a removal singularity at $\infty$ if and only if $f$ is constant. (Prove This!)
- An entire function $f$ has a pole of order $m$ at $\infty$ if and only if $f$ is a polynomial of degree $m$. (Prove This!)
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