derivation of bosonization formula for chiral fermions in 1D

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A complete and explicit confirmation of the bosonization formula on the circle is given by direct construction of the anticommutator of the fermionic and bosonic forms of the fermion field. The action of the unitary "shift" operator $\exp i\theta_R$ on the fermion operators $c_k^\dagger$ is given.

Impose antiperiodic boundary conditions on fermion fields on a circle of perimeter L, so $\exp ikL = -1$. Fermion creation and annihilation operators obey

$$\{c_k, c_{k'}\} = 0, \quad \{c_k, c_{k'}^\dagger\} = \delta_{kk'}. \tag{1}$$

We will work in the right-moving chiral Hilbert space of states $|\psi\rangle$ where

$$\lim_{k\to\infty} c_k|\psi\rangle = 0, \quad \lim_{k\to-\infty} c_k^\dagger|\psi\rangle = 0 \tag{2}$$

Then define

$$N_R = \sum_{k>0} c_k^\dagger c_k - c_{-k}c_{-k}^\dagger. \tag{3}$$

with

$$N_R c_k^\dagger = c_k^\dagger (N_R + 1). \tag{4}$$

Also define the unitary shift operator with the action

$$e^{i\theta_R} c_k^\dagger = c_{k+2\pi/L}^\dagger e^{i\theta_R} \tag{5}$$

Then, in the chiral fermion Hilbert space

$$N_R e^{i\theta_R} = e^{i\theta_R}(N_R + 1). \tag{6}$$

Also define, for $q > 0$, where $\exp iqL = 1$.

$$b_q^\dagger = \left(\frac{2\pi}{qL}\right)^{\frac{1}{2}} \sum_k c_{k+q}^\dagger c_k, \quad b_q = \left(\frac{2\pi}{qL}\right)^{\frac{1}{2}} \sum_k c_k^\dagger c_{k+q}, \tag{7}$$

where

$$[N_R, b_q^\dagger] = [e^{i\theta_R}, b_q^\dagger] = 0, \tag{8}$$

and

$$[b_q, b_{q'}] = 0, \quad [b_q, b_{q'}^\dagger] = \delta_{qq'} \left(\frac{2\pi}{qL}\right) \sum_k (n_{k+q} - n_k) = \delta_{qq'}\,, \tag{9}$$

where $n_k \equiv c_k^\dagger c_k$.

Then

$$[b_q^\dagger, c_k^\dagger] = \left(\frac{2\pi}{qL}\right)^{\frac{1}{2}} c_{k+q}^\dagger, \quad [b_q, c_k^\dagger] = \left(\frac{2\pi}{qL}\right)^{\frac{1}{2}} c_{k-q}^\dagger \tag{10}$$

Now define

$$\Psi_f^\dagger(x) = \frac{1}{\sqrt{L}} \sum_k e^{ikx} c_k^\dagger, \tag{11}$$

and

$$\varphi_R^+(x,\epsilon) = \left(\varphi_R^-(x,\epsilon\right)^\dagger = \pi N_R(x/L) - i\sum_{q>0} f(q\epsilon)\left(\frac{2\pi}{qL}\right)^{\frac{1}{2}} e^{iqx} b_q^\dagger, \tag{12}$$

where $\epsilon > 0$, and $f(u)$ is a real regularization function with the property that $f(0) = 1$, and $f(u) \to 0$ as $u \to \infty$, and decreases exponentially or faster in this limit, so the integral from 0 to ∞ of $u^n f(u)$ is finite for all integers $n \geq 0$.

Then

$$b_q^\dagger \Psi_f^\dagger(x) = \Psi_f^\dagger(x)\left(b_q^\dagger + \left(\frac{2\pi}{qL}\right)^{\frac{1}{2}} e^{-iqx}\right), \quad b_q \Psi_f^\dagger(x) = \Psi_f^\dagger(x)\left(b_q + \left(\frac{2\pi}{qL}\right)^{\frac{1}{2}} e^{iqx}\right). \tag{13}$$

From this

$$\begin{aligned}
e^{\pm i\varphi_R^-(x,\epsilon)}\Psi_f^\dagger(x') &= \exp\pm\left(\frac{i\pi x}{L} - \frac{2\pi}{L}\sum_{q>0}\frac{f(q)}{q}e^{-iq(x-x')}\right)\Psi_f^\dagger(x')e^{\pm i\varphi_R^-(x,\epsilon)} \\
e^{\pm i\varphi_R^+(x,\epsilon)}\Psi_f^\dagger(x') &= \exp\pm\left(\frac{i\pi x}{L} + \frac{2\pi}{L}\sum_{q>0}\frac{f(q)}{q}e^{iq(x-x')}\right)\Psi_f^\dagger(x')e^{\pm\varphi_R^+(x,\epsilon)} \\
e^{\pm i\theta_R}\Psi_f^\dagger(x') &= \exp\left(\mp\frac{2\pi i x'}{L}\right)\Psi_f(x')e^{\pm i\theta_R}.
\end{aligned} \tag{14}$$

Now define the regularized form of the bosonized representation of the fermion field:

$$\Psi_b^\dagger(x,\epsilon) = \frac{1}{\sqrt{L}}e^{i\varphi_R^+(x,\epsilon)}e^{i\theta}e^{i\varphi_R^-(x,\epsilon)} \tag{15}$$

Then

$$\begin{aligned}
\{\Psi_b(x,\epsilon),\Psi_f^\dagger(x')\} &= e^{-i\varphi_R(x,\epsilon)}e^{-i\theta_R}e^{-i\pi(x'/L)}\Psi_f^\dagger(x')e^{-i\varphi^-(x',\epsilon)}\times \\
&\quad\frac{1}{L}\left(\exp F(x-x',\epsilon) + \exp F(x'-x,\epsilon)\right), \qquad (16)\\
\{\Psi_b^\dagger(x,\epsilon),\Psi_f^\dagger(x')\} &= e^{i\varphi_R(x,\epsilon)}e^{i\theta_R}e^{i\pi(x'/L)}\Psi_f^\dagger(x')e^{i\varphi^-(x,\epsilon)}\times \\
&\quad\frac{1}{L}\left(\exp -F(x-x',\epsilon) + \exp -F(x'-x,\epsilon)\right), \qquad (17)
\end{aligned}$$

where

$$F(x,\epsilon) = i\pi(x/L) + \frac{2\pi}{L}\sum_{q>0}\frac{f(q\epsilon)}{q}e^{iqx}. \tag{18}$$

The key requirement on the cutoff function $f(u)$ is that $F(z,\epsilon)$ with $\epsilon > 0$ is holomorphic for $\operatorname{Im} z \geq 0$; this is assured by the properties of $f(u)$ given earlier. Then for $\operatorname{Im} z > 0$,

$$\lim_{\epsilon\to 0^+}\exp F(z,\epsilon) = \frac{i}{2\sin(\pi z/L)}. \tag{19}$$

This is also holomorphic on the real axis, except when $z = nL$, when it diverges at a first-order pole, and is odd around these points, and is antiperiodic on the circle. Then

$$\lim_{\epsilon\to 0^+}\frac{1}{L}\left(e^{F(x,\epsilon)} + e^{F(-x,\epsilon)}\right) = c\sum_n(-1)^n\delta(x-nL) \tag{20}$$

The weight c is given by

$$\tfrac{1}{2}c = \lim_{\epsilon\to 0^+}\int_{-\frac{1}{2}L}^{\frac{1}{2}L} dx\, \exp(F(x,\epsilon)), \tag{21}$$

which can be evaluated by deforming the integration contour from the real axis to a path from $-\frac{1}{2}L$ to $-\frac{1}{2}L + i\infty$ to $\frac{1}{2}L + i\infty$ to $\frac{1}{2}L$, which leaves its value unchanged, as the function is holomorphic in the upper half of the complex plane. Then

$$c = \frac{4}{L}\int_0^\infty dy\, \frac{1}{2\cosh(\pi y/L)} = 1. \tag{22}$$

Thus

$$\lim_{\epsilon\to 0^+} \{\Psi_b(x,\epsilon), \Psi_f^\dagger(x')\} = \hat{O}(x)\sum_n (-1)^n \delta(x - x' + nL) \tag{23}$$

where

$$\hat{O}(x) = e^{-i\varphi_R(x,0)} e^{-i\theta_R} e^{-i\pi(x/L)} \Psi_f^\dagger(x) e^{-i\varphi^-(x,0)}. \tag{24}$$

We now need to show that this commutes with all operators in the chiral-fermion algebra, and is independent of x, so $\hat{O}(x) = \eta$, a constant. Commutation with either c_k, $c_k^\dagger$, or N_R, $e^{i\theta_R}$, b_q and $b_q^\dagger$ is easily established using the previously given commutation relations. Thus a single diagonal matrix element is needed to define η. For the states $|N\rangle = \exp(iN\theta_R)|0\rangle$ that are annihilated by all b_q, and which are eigenvalues of N_R with eigenvalue N,

$$\eta = \langle N|\hat{O}(x)|N\rangle = \langle N+1|c^\dagger_{(2N+1)\pi/L}|N\rangle = \langle 1|c^\dagger_{\pi/L}|0\rangle, \tag{25}$$

which is independent of x, and has the property

$$\eta^*\eta = \langle 0|c_{\pi/L}c^\dagger_{\pi/L}|0\rangle = 1. \tag{26}$$

Then

$$\lim_{\epsilon\to 0^+} \{\Psi_b(x,\epsilon), \Psi_f^\dagger(x')\} = \eta\sum_n (-1)^n \delta(x - x' + nL)\hat{O}(x), \tag{27}$$

where η is the unimodular Klein factor, which commutes with all elements of the chiral fermion algebra. Since η is defined by an off-diagonal matrix element, its phase is fundamentally an arbitrary choice.

In addition,

$$\lim_{\epsilon\to 0^+} (\exp -F(x,\epsilon) + \exp -F(-x,\epsilon)) = 0, \tag{28}$$

Hence

$$\lim_{\epsilon\to 0^+} \{\Psi_b^\dagger(x,\epsilon), \Psi_f^\dagger(x')\} = 0. \tag{29}$$

This demonstrates the operator identity in the chiral fermion Hilbert space:

$$\lim_{\epsilon\to 0^+} \eta\Psi_b^\dagger(x,\epsilon) = \Psi_f^\dagger(x). \tag{30}$$

or for right-moving chiral fermions, with $\varphi_R^\pm(x) \equiv \varphi_R^\pm(x,0)$,

$$\sqrt{L}\Psi^\dagger(x) \equiv \sum_k e^{ikx}c_k^\dagger = \eta e^{i\varphi_R^+(x)} e^{i\theta_R} e^{i\varphi_R^-(x)} = \lim_{\epsilon\to 0^+} \eta e^{\frac{1}{2}F(0,\epsilon)} e^{i\varphi_R(x,\epsilon)} \tag{31}$$

where

$$\varphi_R(x) = (\varphi_R(x))^\dagger = \varphi_R^+(x,\epsilon) + \theta_R + \varphi_R^-(x,\epsilon). \tag{32}$$

If there is more than one species of chiral fermions, the unimodular Klein factors ensure that fermion fields of two different species anticommute.

$$\{\eta_\sigma, c^\dagger_{k,\sigma'}\} = \{\eta_\sigma, c_{k,\sigma'}\} = 0, \quad \sigma \neq \sigma'. \tag{33}$$

If the species labels σ are are ordered so that the Dirac sea of species σ is filled after that of species $\sigma' > \sigma$, the conventional phase choice for the Dirac-sea states corresponds to

$$\eta_\sigma = \prod_{\sigma'<\sigma} (-1)^{N_{\sigma'}}, \tag{34}$$

and $\eta_1 = 1$.

The above treatment has followed the usual treatment with standard bosons b_q, $b_q^\dagger$, with $q > 0$. A useful rational variant, which avoids square roots, is to define A_q (for all q, positive, negative or zero, with $\exp iqL = 1$) by

$$A_0 = N_R; \quad A_q = \sum_k c_{k+q}^\dagger c_k, \quad q \neq 0. \tag{35}$$

Then

$$[A_q, A_{q'}] = \left(\frac{q'L}{2\pi}\right) \delta_{q+q',0}. \tag{36}$$

Also define B_q by

$$B_0 = \tfrac{1}{2}A_0^2 + \sum_{q>0} A_q A_{-q}; \quad B_q = \tfrac{1}{2}\sum_{q'} A_{q-q'}A_{q'}, \quad q \neq 0. \tag{37}$$

Then

$$[B_q, B_q'] = \tfrac{1}{12}\left(\left(\frac{q'L}{2\pi}\right)^3 - \left(\frac{q'L}{2\pi}\right)\right)\delta_{q+q',0} + \left(\frac{(q'-q)L}{2\pi}\right) B_{q+q'}. \tag{38}$$

Note that B_q, with $q = -2\pi n/L$, are the standard generators L_n of the Virasoro algebra with central charge $c = 1$. In this formulation, the charge and momentum densities (both relative to the state $|0\rangle$) are

$$\rho_R(x) \;=\; \frac{1}{L}\sum_q A_q e^{iqx}, \tag{39}$$

$$\pi_R(x) \;=\; \frac{2\pi\hbar}{L^2}\sum_q B_q e^{iqx} \tag{40}$$

and

$$\varphi_R^+(x,\epsilon) = \frac{\pi x}{L}A_0 - \frac{2\pi i}{L}\sum_{q>0}\frac{f(q\epsilon)}{q}e^{iqx}A_q. \tag{41}$$

This form makes the passage to the thermodynamic limit (where the circle of circumference L becomes the infinite line) much easier. In this limit k and q become continuous variables, so

$$\{c(k), c^\dagger(k')\} = 2\pi\delta(k-k'), \tag{42}$$

and

$$A(q) = \int_{k_F}^{\infty}\frac{dk}{2\pi}c^\dagger(k+\tfrac{1}{2}q)c(k-\tfrac{1}{2}q) - \int_{-\infty}^{k_F} c(k-\tfrac{1}{2}q)c^\dagger(k+\tfrac{1}{2}q), \tag{43}$$

so

$$[A(q), A(q')] = q'\delta(q+q'). \tag{44}$$

Note that

$$N_R = A(0) \tag{45}$$

is the particle number operator (relative to the Dirac-sea state with Fermi vector k_F). Then

$$B(q) = \int_0^\infty dq' A(\tfrac{1}{2}q + q')A(\tfrac{1}{2}q - q'), \tag{46}$$

and

$$[B(q), B(q')] = \tfrac{1}{12}(q')^3\delta(q + q') + (q' - q)B(q + q'). \tag{47}$$

The charge and momentum density (relative to a Dirac sea with $k_F = 0$), are

$$\rho(x) = \frac{k_F}{2\pi} + \int_\infty^\infty \frac{dq}{2\pi} A(q)e^{iqx}, \tag{48}$$

$$\pi(x) = \frac{\hbar(k_F)^2}{4\pi} + \hbar\int_{-\infty}^\infty \frac{dq}{2\pi} B(q)e^{iqx}. \tag{49}$$

and

$$\varphi_R(x, \epsilon) = -i\int_{-\infty}^\infty \frac{dq}{q}\, f(|q|\epsilon)e^{iqx}A(q). \tag{50}$$

The decomposition used for normal ordering is now given by

$$\varphi_R(x, \epsilon) = \varphi_R^+(x, \epsilon) + \theta_R + \varphi_R^-(x, \epsilon) \tag{51}$$

where

$$\begin{aligned}
\varphi_R^+(x, \epsilon) &= \lim_{\alpha\to 0^+} -i\int_\alpha^\infty \frac{dq}{q} f(|q|\epsilon)e^{iqx}A(q) \\
\theta_R &= \lim_{\alpha\to 0^+} -i\int_{-\alpha}^\alpha \frac{dq}{q} f(|q|\epsilon)e^{iqx}A(q) \\
\varphi_R^-(x, \epsilon) &= \lim_{\alpha\to 0^+} -i\int_\infty^{-\alpha} \frac{dq}{q} f(|q|\epsilon)e^{iqx}A(q).
\end{aligned} \tag{52}$$

The commutation relation

$$[N_R, e^{i\theta_R}] = e^{i\theta_R} \tag{53}$$

now derives from the commutation relation

$$[A(0), q^{-1}A(q)] = \delta(q). \tag{54}$$

Here the meaning of taking the limit $\alpha \to 0^+$ is not to set $\alpha = 0$, but to obtain the limiting behavior in this limit. In general, the normal-ordered form of any operator will be multiplied by a factor $(\alpha')^\Delta$, where Δ is the scaling dimension of the operator; $\alpha' = \alpha e^{\mathbb{C}}$ (where $\mathbb{C} = 0.577\ldots$ is Euler's constant) is an infra-red regularization that replaces $2\pi/L$ in expressions derived as $L \to \infty$ limits of those on the circle.

Then

$$\Psi^\dagger(x) \equiv \int_{-\infty}^\infty \frac{dk}{2\pi} e^{ikx}c^\dagger(k) = \left(\frac{\alpha'}{2\pi}\right)^{\frac{1}{2}} \eta e^{ik_Fx}e^{i\varphi_R^+(x)}e^{i\theta_R}e^{i\varphi_R^-(x)} = \lim_{\epsilon\to 0^+}\left(\frac{e^{c_f}}{2\pi\epsilon}\right)^{\frac{1}{2}} \eta\, e^{ik_Fx}e^{i\varphi_R(x,\epsilon)} \tag{55}$$

where c_f is the cutoff-function-dependent constant

$$c_f = \lim_{v\to 0^+}\left(\ln v + \mathbb{C} + \int_v^\infty \frac{du}{u} f(u)\right). \tag{56}$$