Notes on the bosonization operator identity for chiral fermions

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(Dated: August 9, 2019 v0.14)

A short summary of the exact fermion-boson "bosonization" duality of free-fermion 1+1d $U(1)$ conformal field theory.

BOSONIZATION ON THE CIRCLE

Consider the right-moving branch of chiral fermions on the circle of circumference L, with antiperiodic boundary conditions:

$$\Psi_R^\dagger(x) = -\Psi_R^\dagger(x+L) = \frac{1}{\sqrt{L}}\sum_k e^{ikx}c_k^\dagger, \quad e^{ikL} = -1. \tag{1}$$

Here x is real, $k = \pm\pi/L, \pm 3\pi/L, \pm 5\pi/L, \ldots, \pm\infty$, and $\{c_k, c_{k'}^\dagger\} = \delta_{kk'}$, with $c_k^\dagger c_k \equiv n_k$. Here

$$\{\Psi_R(x), \Psi_R^\dagger(x')\} = \frac{1}{L}\sum_k e^{ik(x-x')} = \sum_n (-1)^n \delta(x - x' + nL), \tag{2}$$

where the RHS is the antiperiodically-repeated Dirac delta function. Note that $\Psi_R(x)|\psi\rangle$ is not a normalizable state, even if $|\psi\rangle$ is, and the Dirac delta function is a distribution, not a true function. In the usual way, the field operator can be defined by a regularization process, in this case

$$\Psi_R^\dagger(x) = \lim_{\epsilon\to 0^+} \Psi_R^{f\dagger}(x,\epsilon) = \frac{1}{\sqrt{L}}\sum_k e^{-|k|\epsilon} e^{ikx} c_k^\dagger \tag{3}$$

where

$$\{\Psi_R^f(x,\epsilon), \Psi_R^{f\dagger}(x',\epsilon')\} = \frac{i}{2L\sin(\pi(x-x') + i(\epsilon+\epsilon'))/L)} + \frac{i}{2L\sin(\pi(x'-x) + i(\epsilon+\epsilon'))/L)}. \tag{4}$$

Then define number and momentum operators

$$N_R = \sum_{k>0} c_k^\dagger c_k - c_{-k}c_{-k}^\dagger, \tag{5}$$

$$P_R = \hbar\sum_{k>0} k\left(c_k^\dagger c_k + c_{-k}c_{-k}^\dagger\right). \tag{6}$$

Here N_R has integer eigenvalues N, while the eigenvalues of P_R are non-negative, with values $\pi\hbar M/L$ where M is an integer satisfying $(-1)^M = (-1)^N$, $M \geq N^2$. The Hilbert space will be defined as the space spanned by the eigenstates of P_R with finite eigenvalue.

Define the (Virasoro) "primary states" $|N\rangle$ by

$$c_k^\dagger|N\rangle = 0, \quad k < 2\pi N/L; \quad c_k|N\rangle = 0, \quad k > 2\pi N/L. \tag{7}$$

Then

$$N_R|N\rangle = N|N\rangle, \quad |N\rangle = \left(e^{i\theta_R}\right)^N |0\rangle, \tag{8}$$

where $e^{i\theta_R}$ is unitary.

Then, for $q > 0 = 2\pi/L, 4\pi/L, 6\pi/L, \ldots, \infty$, with $e^{iqL} = 1$, define

$$b_q = \sqrt{(2\pi/qL)}\sum_k c_k^\dagger c_{k+q}. \tag{9}$$

These annihilate the primary states

$$b_q|N\rangle = 0, \tag{10}$$

and, as a consequence of the "chiral anomaly" $\sum_k (n_{k+q} - n_k) = qL/2\pi$, obey the Heisenberg (boson) algebra

$$[b_q, b^\dagger_{q'}] = \delta_{qq'}, \quad [b_q, b_{q'}] = 0. \tag{11}$$

In terms of these operators

$$P_R = \hbar \left(\frac{\pi}{L} N_R^2 + \sum_{q>0} q b^\dagger_q b_q \right), \tag{12}$$

$$\Psi^\dagger_R(x) = \frac{1}{\sqrt{L}} e^{i\varphi^+_R(x)} e^{i\theta_R} e^{i\varphi^-_R(x)}, \tag{13}$$

where $\varphi^+_R(x) = (\varphi^-_R(x))^\dagger$, and

$$\varphi^-_R(x) = \pi N_R(x/L) + i \sum_{q>0} \sqrt{(2\pi/qL)} e^{-iqx} b_q. \tag{14}$$

Note that there is no cutoff parameter of any kind in this expression.

Note that the equivalence of fermionic and bosonic forms are *exact operator identities* for chiral fermions in the Hilbert space described above. It is easy to check this with explicit calculations. Consider $\langle \psi_2 | \Psi^\dagger_R(x) | \psi_1 \rangle$ where, for example

$$|\psi_1\rangle = b^\dagger_q |0\rangle, \quad |\psi_2\rangle = |1\rangle.$$

In the fermionic formalism, with $k_0 = \pi/L$,

$$|\psi_1\rangle = \sqrt{(2\pi/qL)} \sum_{-q<k<0} c^\dagger_{k+q} c_k |0\rangle, \quad |\psi_2\rangle = c^\dagger_{k_0} |0\rangle.$$

Then

$$\langle \psi_2 | \Psi^\dagger_R(x) | \psi_1 \rangle = \frac{1}{\sqrt{L}} \sqrt{(2\pi/qL)} \sum_{kk'} e^{ikx} \langle 0 | c_{k_0} c^\dagger_k c^\dagger_{k'+q} c_{k'} | 0 \rangle = -\frac{1}{\sqrt{L}} \sqrt{(2\pi/qL)} e^{i(k_0 - q)x}.$$

In the bosonized form, this is given by

$$\frac{1}{\sqrt{L}} \langle 0 | e^{-i\theta_R} e^{ik_0 x} e^{i\theta_R} (1 + i(i\sqrt{(2\pi/qL)} e^{-iqx} b_q)) b^\dagger_q |0\rangle,$$

which gives the same result. This exercise can be repeated with more complicated states, as the equivalence of the the two forms of $\Psi^\dagger_R(x)$ is an exact mathematical identity.

Now consider the boson-normal-ordered forms of the operators

$$\begin{aligned} \Psi_R(x)\Psi^\dagger_R(x') &= \frac{i}{2L\sin(\pi(x-x')/L)} e^{i(\varphi^+_R(x)-\varphi^+_R(x'))} e^{i(\varphi^-_R(x)-\varphi^-_R(x'))}, \\ \Psi^\dagger_R(x')\Psi_R(x) &= \frac{i}{2L\sin(\pi(x'-x)/L)} e^{i(\varphi^+_R(x)-\varphi^+_R(x'))} e^{i(\varphi^-_R(x)-\varphi^-_R(x'))}. \end{aligned} \tag{15}$$

For $x \neq x'$ (mod L) we get, as required

$$\Psi_R(x)\Psi^\dagger_R(x') + \Psi^\dagger_R(x')\Psi_R(x) = 0, \quad e^{2\pi i(x-x')/L)} \neq 1. \tag{16}$$

Each term diverges as $\exp(2\pi i(x-x')/L \to 1$. In this case, to make sense of the divergence and correctly recover the anticommutator as the Dirac delta function, we must use the "ϵ-regularized" form of one or both of the two field operators:

$$\Psi_R(x,\epsilon) = \frac{1}{\sqrt{L}} e^{-i\varphi^+_R(x+i\epsilon)} e^{-i\theta_R} e^{-i\varphi^-_R(x-i\epsilon)}, \quad \Psi^\dagger_R(x,\epsilon) \equiv (\Psi_R(x,\epsilon))^\dagger = \frac{1}{\sqrt{L}} e^{i\varphi^+_R(x+i\epsilon)} e^{i\theta_R} e^{i\varphi^-_R(x-i\epsilon)}, \tag{17}$$

with real positive ϵ (replace $x \pm i\epsilon$ by $x \mp i\epsilon$ for left-movers). A different ϵ can be used for each fermion field. Note that the "bosonically-regulated" operator $\Psi_R(x,\epsilon)$ is distinct from the "fermionically-regulated" operator $\Psi_R^f(x,\epsilon)$ (3), but they both have the same unregulated limit, so that

$$\lim_{\epsilon\to 0^+} \langle\psi_1|\Psi_R(x,\epsilon)|\psi_2\rangle = \lim_{\epsilon\to 0^+} \langle\psi_1|\Psi_R^f(x,\epsilon)|\psi_2\rangle, \tag{18}$$

where $|\psi_1\rangle$ and $|\psi_2\rangle$ are normalizable states in the Hilbert space spanned by eigenstates of P_R with finite eigenvalue.

The key algebraic result is

$$\exp\left(-[\theta_R + \varphi_R^-(x-i\epsilon), \theta_R + \varphi_R^+(x+i\epsilon')]\right) = 2i\sin\left(\pi((x-x') - i(\epsilon+\epsilon'))/L\right). \tag{19}$$

From this,

$$\begin{aligned}\{\Psi_R(x,\epsilon), \Psi_R^\dagger(x',\epsilon')\} &= \left(\frac{i}{2L\sin(\pi(x-x'+i(\epsilon+\epsilon'))/L)} + \frac{i}{2L\sin(\pi(x'-x+i(\epsilon+\epsilon')/L)}\right) \\ &\times\ e^{i(\varphi_R^+(x+i\epsilon)-\varphi_R^+(x'+i\epsilon'))}e^{i(\varphi_R^-(x-i\epsilon)-\varphi_R^-(x'-i\epsilon'))}.\end{aligned} \tag{20}$$

On taking the limit $\epsilon, \epsilon' \to 0^+$, the RHS becomes equal to the antiperiodically-repeated Dirac delta function $\sum_n (-1)^n \delta(x-x'+nL)$. Similarly,

$$\begin{aligned}\Psi_R^\dagger(x,\epsilon)\Psi_R^\dagger(x',\epsilon') &= \frac{2i}{L}\sin(\pi((x-x')-i(\epsilon+\epsilon'))/L) \times \\ &\quad e^{i(\varphi_R^+(x+i\epsilon)+\varphi_R^+(x'+i\epsilon')}e^{2i\theta_R}e^{i(\varphi_R^-(x-i\epsilon)+\varphi_R^-(x'-i\epsilon')}.\end{aligned} \tag{21}$$

Taking the limit $\epsilon, \epsilon' \to 0^+$ of this normal-ordered operator gives:

$$\{\Psi_R^\dagger(x), \Psi_R^\dagger(x')\} = 0. \tag{22}$$

The density operator is given by

$$\rho_R(x) = \frac{1}{L}\left(N_R + \sum_{q>0}\sqrt{(qL/2\pi)}\left(e^{iqx}b_q^\dagger + e^{-iqx}b_q\right)\right). \tag{23}$$

The normalized state $|\psi_R(x,\epsilon)\rangle \equiv \sqrt{(2L\sinh(2\pi\epsilon/L))}\,\Psi_R^\dagger(x,\epsilon)|0\rangle$ has the periodically-repeated width-2ϵ Lorentzian particle density

$$\langle\psi_R(x,\epsilon)|\rho_R(x')|\psi_R(x,\epsilon)\rangle = \frac{1}{L}\sum_q e^{-|q|\epsilon}e^{iq(x-x')}, \tag{24}$$

where the sum is over all q obeying $\exp iqL = 1$. This becomes a (periodically-repeated) delta function when $\epsilon \to 0^+$.

Bosonization is a strict operator identity only for "complete" chiral fermions without a cutoff on $|k - k_F|$, $k_F = 2\pi N_R/L$, in a Fock space of states where (for right-movers) $n_k \to (1,0)$ as $k \to (-\infty,\infty)$. A cutoff produces non-locality, and equality ("=") gets replaced by long-wavelength proportional equivalence ("$\sim$").

BOSONIZATION ON THE INFINITE LINE

In this case we need to use the regularized form, and undo the boson normal-ordering. On the circle,

$$\begin{aligned}\Psi_R^\dagger(x,\varepsilon) &= \frac{1}{\sqrt{L}}e^{\frac{1}{2}([\theta_R+\varphi_R^+(x+i\epsilon),\theta_R+\varphi_R^-(x-i\epsilon)]}e^{i\varphi_R(x,\alpha)} \\ &= \frac{1}{\sqrt{(2L\sinh(2\pi\epsilon/L)}}e^{i\varphi_R(x,\alpha)}\end{aligned} \tag{25}$$

with

$$\varphi_R(x,\epsilon) = \varphi_R^+(x+i\epsilon) + \theta_R + \varphi_R^-(x-i\epsilon). \tag{26}$$

Note that $\varphi_R(x,\epsilon)$ is Hermitian.

Now take the limit $L \to \infty$, $N_0 \to \infty$, $2\pi N_0/L \to k_{FR}$, with finite ϵ, and $N_R = N_0 + N'_R$, where N'_R is finite.

Then

$$\Psi_R^\dagger(x,\epsilon) = \frac{1}{\sqrt{(4\pi\epsilon)}} e^{i\varphi_R(x,\epsilon)}. \tag{27}$$

with

$$[\varphi_R(x,\epsilon), \varphi_R(x',\epsilon')] = -2i\tan^{-1}\left(\frac{x-x'}{\epsilon+\epsilon'}\right). \tag{28}$$

Also define

$$\rho_R(x,\epsilon) = \frac{1}{2\pi}\partial_x\varphi(x,\epsilon), \tag{29}$$

with

$$[\varphi_R(x,\epsilon), \rho_R(x',\epsilon')] = \frac{i}{\pi}\left(\frac{\epsilon+\epsilon'}{(x-x')^2+(\epsilon+\epsilon')^2}\right). \tag{30}$$

Then

$$\int_{-\infty}^{\infty} dx\, \rho_R(x,\epsilon) = N'_R \tag{31}$$

is independent of ϵ, and has the important property

$$[N'_R, e^{\pm i\varphi_R(x,\epsilon)}] = \pm e^{i\varphi_R(x,\epsilon)}. \tag{32}$$

These commutation relations can be represented in terms of a chiral bosonic field $A_R(q)$ with the Heisenberg algebra commutation relations

$$[A_R(q), A_R(q')] = q'\delta(q+q'), \quad A_R(q)^\dagger = A_R(-q), \tag{33}$$

where

$$\varphi_R(x,\epsilon) = k_{FR}x - i\int_{-\infty}^{\infty}\frac{dq}{q}e^{-|q|\epsilon}e^{iqx}A_R(q), \quad N'_R = A_R(0). \tag{34}$$

A shift of the background ground state charge density to $k_{FR}/2\pi$ has been made, so that the integer N'_R represents a finite number of extra particles added to this background. Note that

$$[N_R, \varphi_R(x,\epsilon)] = -i\int_{-\infty}^{\infty} dq\, e^{-|q|\epsilon}e^{iqx}[A_R(0), q^{-1}A_R(q)] = -i, \tag{35}$$

where $[A_R(0), q^{-1}A_R(q)] = \delta(q)$ has been used. This allows the identification

$$\lim_{\alpha\to 0^+} -i\int_{-\alpha}^{\alpha}\frac{dq}{q}A_R(q) = \theta_R. \tag{36}$$

The singular structure at $q=0$ of the algebra (33) on the infinite line thus reproduces the action-angle variables of the circle geometry.

Primary states (fixed charge ground states) are defined by

$$A_R(0)|N\rangle = N|N\rangle, \quad A_R(q)|N\rangle = 0, \quad q<0. \tag{37}$$

For any operator

$$\widehat{O} = \int_{-\infty}^{\infty} dq\, O(q)A_R(q) \tag{38}$$

there is a decompositon

$$\begin{aligned}\widehat{O} &= \widehat{O}^{+} + \widehat{O}^{0} + \widehat{O}^{-},\\ \widehat{O}^{+} &= \lim_{\alpha\to 0^+}\int_{\alpha}^{\infty} dq\, O(q)A_R(q),\\ \widehat{O}^{0} &= \lim_{\alpha\to 0^+}\int_{-\alpha}^{\alpha} dq\, O(q)A_R(q),\\ \widehat{O}^{-} &= \lim_{\alpha\to 0^+}\int_{-\infty}^{-\alpha} dq\, O(q)A_R(q).\end{aligned} \tag{39}$$

Normal ordering places all occurences of $A_R(q)$ with $q > 0$ to the left of those of $A_R(0)$, and all occurences of $A_R(q)$ with $q < 0$ to the right. In particular,

$$: e^{i\widehat{O}} :\equiv e^{i\widehat{O}^{+}} e^{i\widehat{O}^{0}} e^{i\widehat{O}^{-}}, \tag{40}$$

and

$$e^{i\widehat{O}} = \lim_{\alpha\to 0^+} \exp\left(-\tfrac{1}{2}\int_{\alpha}^{\infty} q dq\, O(q)O(-q)\right) : e^{i\widehat{O}} : . \tag{41}$$

Now consider the normal-ordered form of a product of fermion operators

$$\widehat{O} = \frac{1}{\sqrt{(4\pi\epsilon_1)}} e^{in_1\varphi_R(x_1,\epsilon_1)} \frac{1}{\sqrt{(4\pi\epsilon_2)}} e^{in_2\varphi_R(x_2,\epsilon_2)} \ldots, \tag{42}$$

where $n_i = \pm 1$. First use the BCH formula and the fundamental commutation relation (28) to combine them into a single exponential:

$$\widehat{O} = \left(\prod_i \frac{1}{\sqrt{(4\pi\epsilon_i)}}\right)\left(\prod_{i<j}\left(\frac{(\epsilon_i+\epsilon_j)+i(x_i-x_j)}{\sqrt{((\epsilon_i+\epsilon_j)^2+(x_i-x_j)^2)}}\right)^{n_in_j}\right) e^{i\Theta}, \quad \Theta = \sum_j n_j\varphi_R(x_j,\epsilon_j). \tag{43}$$

Now carry out the normal-ordering:

$$\begin{aligned} e^{i\Theta} &= \lim_{\alpha\to 0^+}\prod_{ij} \exp -n_in_j\left(\tfrac{1}{2}\int_{\alpha}^{\infty}\frac{dq}{q}e^{-q(\epsilon_i+\epsilon_j)}\right)\times\\ &\quad \prod_{i<j}\exp\left(n_in_j\int_0^{\infty}\frac{dq}{q}e^{-q(\epsilon_i+\epsilon_j)}(1-\cos(q(x_i-x_j))\right) : e^{i\Theta} :\\ &= (e^{\mathbb{C}}\alpha)^{\frac{1}{2}(\sum_j n_j)^2}\left(\prod_i\sqrt{(2\epsilon_i)}\right)\left(\prod_{i<j}\left(\sqrt{((x_i-x_j)^2+(\epsilon_i+\epsilon_j)^2)}\right)^{n_in_j}\right) : e^{i\Theta} :,\end{aligned} \tag{44}$$

where $\mathbb{C}$ is Catalan's constant. It is useful to define $\alpha' = \alpha e^{\mathbb{C}}$. Reassembling the parts,

$$\widehat{O} = \lim_{\alpha\to 0^+}(\alpha')^{\frac{1}{2}(\sum_i n_i)^2}\left(\prod_i\frac{1}{\sqrt{2\pi}}\right)\left(\prod_{i<j}((\epsilon_i+\epsilon_j)+i(x_i-x_j))^{n_in_j}\right) : e^{i\Theta} : . \tag{45}$$

This expression corresponds to replacing $2\pi/L$ by α' in the circle-geometry expressions, and taking the limit $\alpha'|x_i - x_j| \ll 1$, with fixed x_i, In these expressions, the formal limit $\alpha \to 0^+$ is not actually taken; instead the limiting power-law behavior as a function of α in this limit is extracted as the scaling dimension of the normal-ordered operator.

Note that provided all the x_i are distinct, the ultra-violet cutoffs ϵ_i have vanished from the prefactor, which depends only on the infra-red cutoff α' as $(\alpha')^{\Delta}$, where $\Delta = \frac{1}{2}(\sum_i n_i)^2$ is the scaling dimension of $\widehat{O}$ as $\alpha \to 0^+$. The absence of the ultraviolet cutoff from the prefactor of the normal ordered form is a very special feature of free fermion systems, where there is no mixing of low-energy physics with processes at the ultra-violet cutoff scale, and the limits $\epsilon_i \to 0^+$ can be taken.

Using this result,

$$\{\Psi^\dagger_R(x,\epsilon),\Psi^\dagger_R(x',\epsilon')\} = \frac{\alpha'}{2\pi}2(\epsilon+\epsilon') : e^{i(\varphi_R(x,\epsilon)+\varphi_R(x',\epsilon'))} : \tag{46}$$

which correctly vanishes when the limits $\epsilon,\epsilon'\to 0^+$ are taken. Similarly

$$\{\Psi_R(x,\epsilon),\Psi^\dagger_R(x',\epsilon')\} = \frac{1}{2\pi}\left(\frac{i}{(x-x')+i(\epsilon+\epsilon')}+\frac{i}{(x'-x)+i(\epsilon+\epsilon')}\right) : e^{i(\varphi_R(x',\epsilon')-\varphi_R(x,\epsilon))} : \tag{47}$$

becomes equal to $\delta(x-x')$ when the limits $\epsilon,\epsilon'\to 0^+$ are taken.

In the Luttinger liquids, once couplings

$$H' = g\int dx \rho_R(x)\rho_L(x) \tag{48}$$

are introduced between fields with opposite chirality, the scaling dimensions of the fermion operators depend on g and ultra-violet cutoffs appear in the prefactor. This is because the "engineering dimension" of the fermion field is $\frac{1}{2}$, and to keep the "engineering dimension" of the operator fixed as its scaling dimension $\Delta(g)$ (the power of α in the prefactor of their boson-normal-ordered forms) changes with g, a counterterm $\epsilon^{\Delta(g)-\frac{1}{2}}$ must also be present in the prefactor.