1. A sphere of radius R carries a charge density $\rho(r) = kr$ (where k is a constant). Find the energy of the configuration. [4]

2. A point charge q of mass m is released from rest at a distance d from an infinite grounded conducting plane. How long will it take for the charge to hit the plane? [3]

3. Consider a charge free volume V bounded by a closed surface S that consists of several separate surfaces (Conductors) S_i each held at potential V_i. Let $\Psi(\mathbf{x})$ be a well behaved function in V and on S, with a value equal to V_i on each surface S_i, but otherwise arbitraty for the present. Define the quantity

$$W[\Psi] = \frac{1}{8\pi}\int_V |\nabla\Psi|^2 d^3x$$

Prove using variational calculus:
$W[\Psi]$, which is nonnegative by definition, is stationary and an absolute minimum if and only if Ψ satisfies the Laplace equation in V and takes on the specified values V_i on the surfaces S_i. [3]

Answers

Answer 1 The electric field for the given charge density can be found by the Gauss law

$$\mathbf{E} = \begin{cases} \dfrac{kr^2}{4\epsilon_0}\hat{\mathbf{r}} & r < R \\ \dfrac{kR^4}{4\epsilon_0 r^2}\hat{\mathbf{r}} & r > R \end{cases} \tag{1}$$

Hence the energy is given by

$$W = \frac{\epsilon_0}{2}\int_{\text{all space}} |\mathbf{E}|^2\, dv \tag{2}$$

$$= \frac{\pi k^2}{8\epsilon_0}\left[\int_0^R r^6 dr + \int_R^\infty \frac{R^8 dr}{r^2}\right] \tag{3}$$

$$= \frac{\pi k^2 R^7}{7\epsilon_0} \tag{4}$$

Alternatively

The Potential at any $r < R$ is given by

$$\phi(r) = \frac{k}{3\epsilon_0\left(R^3 - \frac{r^3}{4}\right)} \tag{5}$$

And then the energy is given by

$$\begin{aligned} W &= \frac{1}{2}\int_0^R \rho(r)\phi(r)4\pi r^2\,dr && (6)\\ &= \frac{4\pi k^2}{6\epsilon_0}\left(R^3\int_0^R r^3\,dr - \frac{1}{4}\int_0^R r^6\,dr\right) && (7)\\ &= \frac{\pi k^2 R^7}{7\epsilon_0} && (8)\end{aligned}$$

Answer 2 From the method of images, the force on charge q is

$$\mathbf{F} = \frac{q^2(-\hat{\mathbf{i}})}{4\pi\epsilon_0(2x)^2} \tag{9}$$

By Newton's Law,

$$\begin{aligned} m\ddot{x} &= -\frac{q^2}{4\pi\epsilon_0(2x)^2} && (10)\\ \dot{x} &= \left(\frac{q^2}{8\pi\epsilon_0 m}\right)^{1/2}\left(\frac{1}{x}-\frac{1}{d}\right)^{1/2} && (11)\\ & && (12)\end{aligned}$$

Integrating further, we get,

$$T = \frac{\pi d}{q}(2\pi\epsilon_0 m d)^{1/2} \tag{13}$$

Answer 3 For a small change in function ψ, the change in the functional W is given by (to the first order)

$$\begin{aligned} \delta W &= W[\psi+\delta\psi] - W[\psi] && (14)\\ &= \frac{1}{4\pi}\oint_S \delta\psi\,\nabla\psi\cdot ds - \frac{1}{8\pi}\int \nabla^2\psi\delta\psi\,dv && (15)\\ & && (16)\end{aligned}$$

The surface integral vanishes, since $\delta\psi = 0$ on surface. Hence δW will vanish for all $\delta\psi$ *if and only if* $\nabla^2\psi = 0$.