Here are some practice problems in vector calculus and curvilinear coordinates.

1. Find equations for the tangent plane and normal line to the surface \( z = x^2 + y^2 \) at the point \((2, -1, 5)\).

2. Find the unit outward normal to the surface \((x - 1)^2 + y^2 + (z + 2)^2 = 9\) at the point \((3, 1, -4)\).

3. Find the divergence and curl of \( \hat{r}/r^2 \).

4. Prove:
   \[
   \begin{align*}
   (a) \quad & \nabla \cdot (\phi \mathbf{A}) = (\nabla \phi) \cdot \mathbf{A} + \phi(\nabla \cdot \mathbf{A}). \\
   (b) \quad & \nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}.
   \end{align*}
   \]

5. Show that \( \mathbf{F} = (2xy + z^3)\mathbf{i} + x^2\mathbf{j} + 3xz^2\mathbf{k} \) is a conservative force field. Find the scalar potential. Find the work done in moving an abject in this field from \((1, -2, 1)\) to \((3, 1, 4)\).

6. Given \( \phi = 2xyz^2 \) and a curve \( C(t) = (t^2, 2t, t^3) \) from \( t = 0 \) to \( t = 1 \). Find \( \int_C \phi \, \mathbf{dr} \).

7. Evaluate \( \int_S \mathbf{A} \cdot d\mathbf{S} \) where \( \mathbf{A} = 18\mathbf{z} - 12\mathbf{j} + 3\mathbf{y}\mathbf{k} \) and \( S \) is that part of the plane \( 2x + 3y + 6z = 12 \) which is located in the first octant.

8. Prove Green’s theorem in a plane. (See any textbook)

9. For a given \( R > 0 \), define \( r_a = \left( x^2 + y^2 + (z - R/2)^2 \right)^{1/2} \) and \( r_b = \left( x^2 + y^2 + (z + R/2)^2 \right)^{1/2} \). The prolate ellipsoidal coordinates are defined as

\[
\begin{align*}
\xi &= \frac{1}{R} (r_a + r_b) \\
\eta &= \frac{1}{R} (r_a - r_b) \\
\phi &= \tan^{-1}\left( \frac{y}{x} \right)
\end{align*}
\]

Find inverse transformations, basis vectors \( e_\xi, e_\eta, e_\phi \), scale factors \( h_\xi, h_\eta, h_\phi \), differential vector \( d\mathbf{r} \), length element \( ds^2 \) and volume element \( dv \).

10. Let

\[
\delta_n(x) = \begin{cases} 
0 & x < -\frac{1}{2n} \\
n & -\frac{1}{2n} < x < \frac{1}{2n} \\
0 & \frac{1}{2n} < x
\end{cases}
\]

Show that

\[
\lim_{n \to \infty} \int_{-\infty}^{\infty} f(x) \delta_n(x) \, dx = f(0)
\]

assuming that the function \( f \) is continuous at \( x = 0 \).
Some Answers:

1. Equation of the tangent plane: 
   \[ -4x + 2y + z = 5 \]
   Equation of the normal line:
   \[ \frac{x - 2}{-4} = \frac{y + 1}{2} = \frac{z - 5}{1} \]

2. Unit outward normal vector: \((2/3, 1/3, -2/3)\).

3. \( \nabla \times \left( \frac{\mathbf{r}}{r^3} \right) = 0 \) and \( \nabla \cdot \left( \frac{\mathbf{r}}{r^3} \right) = 4\pi \delta(\mathbf{r}) \). Proof: Show by explicit differentiation that if \( r \neq 0 \) then \( \nabla \cdot \left( \frac{\mathbf{r}}{r^3} \right) = 0 \). Now choose \( \epsilon > 0 \). Let \( S \) be the spherical surface of radius \( \epsilon \) enclosing the volume \( V \). Then,
   \[
   \int_{V} \nabla \cdot \left( \frac{\mathbf{r}}{r^3} \right) \, dv = \int_{S} \frac{\mathbf{r}}{r^3} \cdot d\mathbf{s} = \int d\Omega = 4\pi
   \]

4. Explicitly differentiate to prove these results.

5. Since \( \nabla \times \mathbf{F} = 0 \), \( \mathbf{F} \) is conservative. Scalar potential is given by
   \[
   \phi(x, y, z) = -\int \mathbf{F} \cdot d\mathbf{r}
   \]
   \[
   = -\int F_x \, dx + \int (\text{Terms in } y \text{ and } z \text{ only}) \, dy + \int (\text{Terms in } z \text{ only}) \, dz
   \]
   \[
   = -(x^2 y + xz^3)
   \]
   Thus work done from \((1, -2, 1)\) to \((3, 1, 4)\) is \( \phi(1, -2, 1) - \phi(1, -2, 1) = -202 \).

6. \( \int \phi \, dr = (8/11, 8/10, 1) \)

7. Note this general procedure: Suppose the given surface, say \( \phi(x, y, z) = c \), has a one-one projection on some domain \( D \) in \( z = 0 \) plane then the surface can also be expressed by equation \( z = f(x, y) \). In this case,
   \[
   \int_{S} \mathbf{A} \cdot d\mathbf{s} = \int_{D} \mathbf{A} \cdot \hat{n} \frac{dx \, dy}{|k \cdot \hat{n}|}
   \]
   Where \( \hat{n} \) is the unit normal to the surface. For our example, (See diagram) \( S \) is the triangular section of the of the plane in the first octant, \( D \) is its projection on \( XY \) plane and is bounded by \( x \) axis, \( y \) axis and the line \( 2x + 3y = 12 \). \( \hat{n} = (2, 3, 6)/7 \). Thus required integral is 24.

8. See Arfken.

9. For a given \( R > 0 \), define \( r_a = \left( x^2 + y^2 + \left( z - R/2 \right)^2 \right)^{1/2} \) and \( r_b = \left( x^2 + y^2 + \left( z + R/2 \right)^2 \right)^{1/2} \)
   The prolate ellipsoidal coordinates are defined as
   \[
   \xi = \frac{1}{R} \left( r_a + r_b \right)
   \]
   \[
   \eta = \frac{1}{R} \left( r_a - r_b \right)
   \]
   \[
   \phi = \tan^{-1} \left( \frac{y}{x} \right)
   \]

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The inverse transformations are given by

\[ x = \frac{R}{2} \sqrt{(\xi^2 - 1)(1 - \eta^2)} \cos \phi \]
\[ y = \frac{R}{2} \sqrt{(\xi^2 - 1)(1 - \eta^2)} \sin \phi \]
\[ z = -\frac{R}{2} \xi \eta \]

The differentials are related by

\[
\begin{pmatrix}
    dx \\
    dy \\
    dz
\end{pmatrix} =
\begin{pmatrix}
    \left(\frac{R}{2}\right) \xi \alpha \cos \phi & -\left(\frac{R}{2}\right) \eta \alpha^{-1} \cos \phi & -\left(\frac{R}{2}\right) \beta \sin \phi \\
    \left(\frac{R}{2}\right) \xi \alpha \sin \phi & -\left(\frac{R}{2}\right) \eta \alpha^{-1} \sin \phi & \left(\frac{R}{2}\right) \beta \cos \phi \\
    -\left(\frac{R}{2}\right) \eta & -\left(\frac{R}{2}\right) \xi & 0
\end{pmatrix}
\begin{pmatrix}
    d\xi \\
    d\eta \\
    d\phi
\end{pmatrix}
\]

where \( \alpha = \sqrt{\frac{1 - \eta^2}{\xi^2 - 1}} \) and \( \beta = \sqrt{(\xi^2 - 1)(1 - \eta^2)} \).

The basis vectors are given by the three columns of the matrix given above. The system is orthogonal. The differential vector is given by

\[ ds = \left(\frac{R}{2}\right) \sqrt{(\xi^2 - \eta^2)/(\xi^2 - 1)} e_\xi d\xi + \left(\frac{R}{2}\right) \sqrt{(\xi^2 - \eta^2)/(1 - \eta^2)} e_\eta d\eta + \left(\frac{R}{2}\right) \sqrt{(\xi^2 - 1)(1 - \eta^2)} e_\phi d\phi \]

The volume element is given by

\[ dv = \left(\frac{R}{2}\right)^3 (\xi^2 - \eta^2) d\xi d\eta d\phi \]

10. Let

\[ S_n = \int_{-\infty}^{\infty} f(x) \delta_n(x) \, dx = n \int_{-1/n}^{1/n} f(x) \, dx \]

Let \( \epsilon > 0 \) be any infinitesimally small number. Since \( f \) is continuous at \( x = 0 \), \( \exists \delta > 0 \) such that \( \forall |x| < \delta, |f(x) - f(0)| < \epsilon \). Now choose an integer \( N \) such that \( \frac{1}{2N} < \delta \).

Then, for every \( n > N \), clearly, \( |x| < \frac{1}{2n} \Rightarrow |x| < \frac{1}{2n} < \frac{1}{2N} < \delta \)

\[ |S_n - f(0)| = n \int_{-1/2n}^{1/2n} |f(x)| \, dx \]
\[ = n \int_{-1/2n}^{1/2n} |f(x) - f(0)| \, dx \]
\[ \leq n \int_{-1/2n}^{1/2n} |f(x) - f(0)| \, dx \]
\[ \leq n \cdot \frac{1}{n} \cdot \epsilon = \epsilon \]

Thus, \( S_n \to f(0) \) as \( n \to \infty \). Here it is also true that for a given \( x \neq 0 \), \( \delta_n(x) \to \delta(x) \) as \( n \to 0 \). Remember, such pointwise convergence is not necessary in the definition of Dirac delta function. For example,

\[ \delta_n(x) = \frac{\sin nx}{\pi x} \]

converges to delta function as \( n \to \infty \) (that is \( S_n \to f(0) \)), however for a given \( x \), \( \delta_n(x) \) is oscillatory.