1. **Verify, by direct substitution, that $G_\pm = e^{\pm ikr}/r$ are solutions of**

$$\left(\nabla^2 + k^2\right) G(\mathbf{r}) = -4\pi\delta(\mathbf{r}).$$

2. **Show that**

$$k|\vec{r} - \vec{r'}| = kr - k(\hat{r}\cdot\vec{r'}) + \frac{k(\hat{r}\times\vec{r'})^2}{2r} + \cdots.$$

Solution is

$$\begin{aligned}
|\bar{r} - \bar{r}'| &= r\left(1 + \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\hat{r}\cdot\hat{r}'\right)^{1/2} \\
&= r\left(1 - \left(\frac{r'}{r}\right)\hat{r}\cdot\hat{r}' + \frac{1}{2}\left(\frac{r'}{r}\right)^2 - \frac{1}{2}\left(\frac{r'}{r}\right)^2 (\hat{r}\cdot\hat{r}')^2 + \cdots\right) \\
&= r - \hat{r}\cdot\bar{r}' + \frac{(\hat{r}\times\bar{r}')^2}{2r} + \cdots
\end{aligned}$$

since

- $(\hat{r}\times\hat{r}')^2 = \sin^2\theta = 1 - \cos^2\theta = 1 - (\hat{r}\cdot\hat{r}')^2$
- $(1+x)^{1/2} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + O\left(x^5\right)$

3. **Show that the gaussian wave packet moves without appreciable change in the width over time t if $t << 2m/\hbar(\Delta k)^2$.** The Gaussian wave packet is given by

$$|\Psi(x,t)|^2 = \frac{1}{\sqrt{\pi}\sigma(t)} \exp\left[-\frac{\left(x - \frac{\hbar k_0}{m}t\right)^2}{\sigma^2(t)}\right]$$

where $\sigma(t) = \sigma_0\left(1 + \frac{\hbar^2}{m^2\sigma_0^4}t^2\right)$ indicates the spread in the wave packet at time t. If $t << \frac{m\sigma_0^2}{\hbar}$, then the spread does not change appreciably. And $\Delta k = \frac{\sqrt{2}}{\sigma_0}$.

4. **Apply the Born approximation to obtain differential cross section for the following potentials:**

(a) **The square well potential**

$$\begin{aligned}
V(r) &= -V_0 \quad \text{for} \quad r < a && (1)\\
&= 0 \quad \text{for} \quad r > a && (2)
\end{aligned}$$

First we find $V(\mathbf{q})$.

$$\begin{aligned}
V(\mathbf{q}) &= \frac{1}{(2\pi)^3}\int d^3r V(\mathbf{r})\, e^{i\mathbf{q}\cdot\mathbf{r}} \\
&= \frac{1}{(2\pi)^3}\int r^2 dr V(r) \int_0^\pi e^{iqr\cos\theta}\sin\theta d\theta \\
&= \frac{1}{(2\pi)^3}\frac{2}{q}\int r dr V(r)\sin(qr) \\
&= -\frac{1}{(2\pi)^3}\frac{2V_0}{q}\int_0^a r\sin(qr)\, dr \\
&= \frac{1}{(2\pi)^3}\frac{2V_0}{q^3}(aq\cos aq - \sin aq)
\end{aligned}$$

Then the differential cross section

$$\frac{d\sigma}{d\Omega} = \frac{m^2 V_0^2}{\pi^2 \hbar^2 q^6} (aq \cos aq - \sin aq)^2$$

where $q = 2k \sin(\Theta/2)$

(b) **The Gaussian Potential**

$$V(r) = -V_0 \exp\left[-\frac{1}{2}\left(\frac{r}{a}\right)^2\right]$$

To find $V(\mathbf{q})$,

$$\begin{aligned} V(\mathbf{q}) &= \frac{1}{(2\pi)^3} \int d^3 r V(\mathbf{r}) e^{i\mathbf{q}\cdot\mathbf{r}} \\ &= \frac{1}{(2\pi)^3} \frac{2}{q} \int r dr V(r) \sin(qr) \\ &= -\frac{1}{(2\pi)^3} \frac{2V_0}{q} \int_0^\infty r \exp\left(-\frac{1}{2}\left(\frac{r}{a}\right)^2\right) \sin(qr)\, dr \\ &= -\frac{1}{(2\pi)^3} \frac{2V_0}{q} \frac{1}{2i} \int_{-\infty}^\infty r e^{-\frac{1}{2}\frac{r^2}{a^2} + iqr} dr \\ &= -\frac{1}{(2\pi)^3} 2\sqrt{\pi} V_0 a^3 e^{-\frac{1}{2}a^2 q^2} \end{aligned}$$

The scattering amplitude is given by

$$f_k(\hat{r}) = \frac{m V_0 a^3}{\pi \hbar^2} \sqrt{\pi} e^{-\frac{1}{2}a^2 q^2}$$

And hence the differenetial cross section is given by

$$= \frac{m^2 V_0^2 a^6}{\pi \hbar^4} e^{-a^2 q^2}$$

(c) **The Exponential Potential**

$$V(r) = -V_0 \exp\left(-\frac{r}{a}\right)$$

To find $V(\mathbf{q})$,

$$\begin{aligned} V(\mathbf{q}) &= \frac{1}{(2\pi)^3} \int d^3 r V(\mathbf{r}) e^{i\mathbf{q}\cdot\mathbf{r}} \\ &= \frac{1}{(2\pi)^3} \frac{2}{q} \int r dr V(r) \sin(qr) \\ &= -\frac{1}{(2\pi)^3} \frac{2V_0}{q} \int_0^\infty r \exp\left(-\frac{r}{a}\right) \sin(qr)\, dr \\ &= -\frac{1}{(2\pi)^3} \frac{4V_0}{q} \frac{a^3 q}{1 + 2a^2 q^2 + a^4 q^4} \end{aligned}$$

Plot the differential cross section in each case.

5. **The scattering of fast electrons by a complex atom can be, in many cases, represented fairly accurately by the following form for the potential energy distribution:**

$$V = -\frac{Ze^2}{r} + Ze^2 \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3 r'$$

For the hydrogen atom in ground state, we may write

$$\rho(r) = |\psi_{1s}|^2$$

Calculate differential cross section.

Suppose a charge distribution is given by $\rho\left(\mathbf{r}\right)$, then the potential energy of another charge Q due to potential of $\rho\left(r\right)$ is

$$V\left(\mathbf{r}\right) = Q\int \frac{\rho\left(\mathbf{r}'\right)}{\left|\mathbf{r}-\mathbf{r}'\right|} d^3r'$$

The the Fourier transform of V

$$\begin{aligned}
V\left(\mathbf{q}\right) &= \int V\left(\mathbf{r}\right) e^{-i\mathbf{q}\cdot\mathbf{r}} d^3r \\
&= Q\int\int \frac{\rho\left(\mathbf{r}'\right)}{\left|\mathbf{r}-\mathbf{r}'\right|} e^{-i\mathbf{q}\cdot\mathbf{r}} d^3r' d^3r \\
&= Q\int \rho\left(\mathbf{r}'\right) d^3r' \int \frac{e^{-i\mathbf{q}\cdot\mathbf{r}}}{\left|\mathbf{r}-\mathbf{r}'\right|} d^3r \\
&= \frac{4\pi Q}{q^2}\int \rho\left(\mathbf{r}'\right) e^{-i\mathbf{q}\cdot\mathbf{r}'} d^3r' \\
&= \frac{4\pi Q}{q^2}\rho\left(\mathbf{q}\right)
\end{aligned}$$

Now since $\left|\psi_{1s}\left(r\right)\right|^2 = \frac{1}{\pi a^3}\exp\left(-2r/a\right)$. The net charge distribution for this problem is given by

$$\rho\left(\mathbf{r}\right) = -\frac{Ze}{r}\delta\left(\mathbf{r}\right) + \frac{Ze}{\pi a^3} e^{-2r/a}$$

Now we only need to calculate Fourier transform of this.