Prolate Ellipsoidal Coordinate System

For a given $R > 0$, define $r_a = \left(x^2 + y^2 + (z - R/2)^2\right)^{1/2}$ and $r_b = \left(x^2 + y^2 + (z + R/2)^2\right)^{1/2}$. The prolate ellipsoidal coordinates are defined as

$$\begin{aligned} \xi &= \frac{1}{R}(r_a + r_b) \\ \eta &= \frac{1}{R}(r_a - r_b) \\ \phi &= \tan^{-1}\left(\frac{y}{x}\right) \end{aligned}$$

The inverse transformations are given by

$$\begin{aligned} x &= \frac{R}{2}\sqrt{(\xi^2 - 1)(1 - \eta^2)}\cos\phi \\ y &= \frac{R}{2}\sqrt{(\xi^2 - 1)(1 - \eta^2)}\sin\phi \\ z &= -\frac{R}{2}\xi\eta \end{aligned}$$

The differentials are related by

$$\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \begin{pmatrix} \left(\frac{R}{2}\right)\xi\alpha\cos\phi & -\left(\frac{R}{2}\right)\eta\alpha^{-1}\cos\phi & -\left(\frac{R}{2}\right)\beta\sin\phi \\ \left(\frac{R}{2}\right)\xi\alpha\sin\phi & -\left(\frac{R}{2}\right)\eta\alpha^{-1}\sin\phi & \left(\frac{R}{2}\right)\beta\cos\phi \\ -\left(\frac{R}{2}\right)\eta & -\left(\frac{R}{2}\right)\xi & 0 \end{pmatrix} \begin{pmatrix} d\xi \\ d\eta \\ d\phi \end{pmatrix}$$

where $\alpha = \sqrt{\frac{1-\eta^2}{\xi^2-1}}$ and $\beta = \sqrt{(\xi^2 - 1)(1 - \eta^2)}$

The basis vectors are given by the three columns of the matrix given above. The system is orthogonal. The differential vector is given by

$$ds = \left(\frac{R}{2}\right)\sqrt{\frac{(\xi^2 - \eta^2)}{(\xi^2 - 1)}}\mathbf{e}_\xi d\xi + \left(\frac{R}{2}\right)\sqrt{\frac{(\xi^2 - \eta^2)}{(1 - \eta^2)}}\mathbf{e}_\eta d\eta + \left(\frac{R}{2}\right)\sqrt{(\xi^2 - 1)(1 - \eta^2)}\mathbf{e}_\phi d\phi$$

The volume element is given by

$$dv = \left(\frac{R}{2}\right)^3 \left(\xi^2 - \eta^2\right) d\xi d\eta d\phi$$