1. Consider a particle trapped in a cubical box given by potential

$$V(x,y,z) = \begin{cases} 0 & 0 \le x, y, z \le L \\ \infty & \text{otherwise.} \end{cases}$$

   (a) Let $n(E)$ be the number of energy eigenstates with energy less than E. Find $n(E)$.

   (b) Find the density of states, which is defined as

$$g(E) = \frac{1}{L^3}\frac{dn}{dE}(E).$$

   (c) Sketch $g(E)$.

2. An anisotropic harmonic oscillator has the potential energy function

$$V(x,y,z) = \frac{1}{2}m\omega^2\left(x^2+y^2\right) + \frac{1}{2}m\omega_z^2 z^2.$$

   (Assume that ω_z/ω is irrational is large.)

   (a) Write down first few eigen-energies and their degeneracies.

   (b) This problem is separable in cartesian coordinates. Write down the eigenfunctions corresponding to the first few eigenstates.

   (c) Do the operators $\mathbf{L}_x$, $\mathbf{L}_y$ and $\mathbf{L}_z$ commute with the hamiltonian? And does $\mathbf{L}^2$?

   (d) Write down the ground state. Is this eigenfunction $\mathbf{L}_z$? If so, what is the eigenvalue?

   (e) The degeneracy of the first excited state (energy: $\frac{1}{2}\hbar\omega_z + 2\hbar\omega$) is two. When separated in cartesian coordinates, the *un-normalized* eigenfunctions are

$$\begin{aligned} \phi_{100}(x,y,z) &= x\exp\left[-\frac{\alpha^2x^2}{2}\right]\exp\left[-\frac{\alpha^2y^2}{2}\right]\exp\left[-\frac{\alpha_z^2z^2}{2}\right] \\ \phi_{010}(x,y,z) &= y\exp\left[-\frac{\alpha^2x^2}{2}\right]\exp\left[-\frac{\alpha^2y^2}{2}\right]\exp\left[-\frac{\alpha_z^2z^2}{2}\right] \end{aligned}$$

   where $\alpha = \sqrt{m\omega/\hbar}$ and $\alpha_z = \sqrt{m\omega_z/\hbar}$. Show that these functions are not eigenfunctions of $\mathbf{L}_z$? Can you construct linear combinations of ϕ_{100} and ϕ_{010}, which are eigenfunctions of $\mathbf{L}_z$? (Hint: Write these functions in spherical polar coordinates and remember $e^{im\phi}$ are eigenfunctions of $\mathbf{L}_z$.)

3. Let $x_1 = x$, $x_2 = y$, and $x_3 = z$. Simillary, for any vector quantity $\mathbf{A}$, let $A_1 = A_x$, $A_2 = A_y$ and $A_3 = A_z$.

   (a) Prove that $\mathbf{L}$ is a hermitian operator.

   (b) Prove $[L_i, x_j] = i\hbar\sum_{k=1}^{3}\epsilon_{ijk}x_k$. Here ϵ_{ijk} is called Levi-Civita antisymmetric symbol, given by

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } ijk = 123, 231, 312 \\ -1 & \text{if } ijk = 132, 321, 213 \\ 0 & \text{otherwise.} \end{cases}$$

   (c) Prove $[L_i, p_j] = i\hbar\sum_{k=1}^{3}\epsilon_{ijk}p_k$.

4. For Legendre polynomials, Prove:

   (a) Orthogonality:
$$\int_{-1}^{1} P_m(x)P_n(x) = \frac{2}{2n+1}\delta_{m,n}$$

   (b) Recursion Relations:
$$\begin{aligned}(n+1)P_{n+1}(x) &= (2l+1)xP_n(x) - nP_{n-1}(x)\\ (1-x^2)\frac{dP_n}{dx} &= -nxP_n(x) + nP_{n-1}(x)\end{aligned}$$

5. Let $T_a = \exp[-ia\hat{P}_x/\hbar]$, where $\hat{P}_x$ is the x-component of the momentum operator.

   (a) If $f' = T_a f$, show that f' is a function obtained by shifting f by a displacement a, that is $f'(x) = f(x-a)$.

   (b) Let $\hat{H} = \frac{1}{2m}\hat{P}_x^2 + V(x)$ be the hamiltonian operator. Show that $\left[\hat{H}, T_a\right] = 0$ if $V(x) = V(x+a)$ for all x.

   (c) If $U_\alpha = \exp[-i\alpha L_z/\hbar]$ is an operator on $L_2(R^3)$, show that
$$U_\alpha f(r,\theta,\phi) = f(r,\theta,\phi+\alpha),$$
   where r, θ and ϕ are spherical polar coordinates.

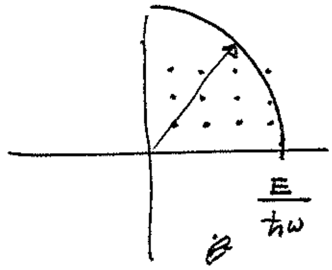
Tutorial 6.

1.(a) The eigenenergies are given by $\hbar\omega = \frac{\hbar^2 \pi^2}{2mL^2}$

$$E_{ijk} = \hbar\omega\left(i^2+j^2+k^2\right). \qquad 0 < i,j,k, \text{ are integers.}$$

Now, ~~for each~~ consider a set of points in $\mathbb{R}^3$.

$$S = \{ (i,j,k) \,/\, 0 \leq i,j,k, \text{ integers} \}$$



For each point in S, there is one energy eigenstate. Thus,

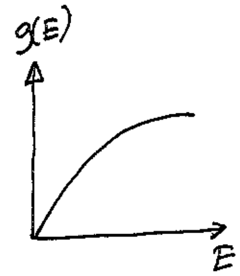
$$n(E) = \Big(\text{number of points of } S, \text{ which lie between Co-ordinate planes and spherical surface given by eq. } i^2+j^2+k^2 = \frac{E}{\hbar\omega}\Big)$$

$$= \frac{1}{8} \text{ Volume of sphere of radius} \left(\frac{E}{\hbar\omega}\right)^{1/2} \qquad \text{For large } E$$

$$= \frac{1}{8} \cdot \frac{4}{3}\pi \left(\frac{E}{\hbar\omega}\right)^{3/2}$$

$$= \frac{\pi}{6} \cdot \left(\frac{2mL^2}{\hbar^2\pi^2}\right)^{3/2} E^{3/2}$$

(b) $$g(E) = \frac{1}{V}\frac{dn}{dE} = \frac{\pi}{6}\left(\frac{2m}{\hbar^2\pi^2}\right)^{3/2} \cdot \frac{3}{2} E^{1/2}$$



2. (a) The eigenenergies are given by

$$E = \hbar\omega\,(i+j+1) + \hbar\omega_z\left(k+\tfrac{1}{2}\right)$$

E	g : degeneracy.
$\frac{1}{2}\hbar\omega_z + \hbar\omega$	1
$\frac{1}{2}\hbar\omega_z + 2\hbar\omega$	2
$\frac{1}{2}\hbar\omega_z + 3\hbar\omega$	3
$\frac{1}{2}\hbar\omega_z + 4\hbar\omega$	4

(b)

(ijk)	Wave fn
(000)	$N_0 N_0 N_0' \cdot \exp(-\frac{\alpha^2}{2}x^2)\exp(-\frac{\alpha^2}{2}y^2)\exp(-\alpha_z^2 z^2/2)$
(100)	$N_1 N_0 N_0' (2\alpha x)\exp[-\frac{\alpha^2}{2}(x^2+y^2) - \frac{\alpha_z^2}{2}z^2]$
(010)	$N_0 N_1 N_0' (2\alpha y)\exp[-\frac{\alpha^2}{2}(x^2+y^2) - \frac{\alpha_z^2}{2}z^2]$
(200)	$N_2 N_0 N_0' (4\alpha^2x^2-2)\exp[-\frac{\alpha^2}{2}(x^2+y^2) - \frac{\alpha_z^2}{2}z^2]$.
(110)	$N_1 N_1 N_0' (4\alpha^2 xy)\exp[-\frac{\alpha^2}{2}(x^2+y^2) - \frac{\alpha_z^2}{2}z^2]$

where $\alpha = \sqrt{\frac{m\omega}{\hbar}}$, $\alpha_z = \sqrt{\frac{m\omega_z}{\hbar}}$, $N_n = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}$

$$N_n' = \left(\frac{m\omega_z}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}$$

(c) Now, $\hat{H} = +\frac{p^2}{2m} + V(\vec{r})$.

Since
$$\frac{p^2}{2m} = -\frac{\hbar^2}{2m}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2}L^2\right)$$

And that $\vec{L}$ is function of θ, ϕ only $\Rightarrow \left[\frac{p^2}{2m}, \vec{L}\right] = 0$

Potential energy operator

$$V(r) = \frac{1}{2}m\omega^2(x^2+y^2) + \frac{1}{2}m\omega_z^2 z^2$$

$$= \frac{1}{2}m\omega^2 r^2\sin^2\theta + \frac{1}{2}m\omega_z^2 r^2\cos^2\theta. \qquad (\omega_z \neq \omega)$$

The operator $L_x = (-i\hbar)\left[-\sin\phi\frac{\partial}{\partial\theta} - \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right]$ contains $\frac{\partial}{\partial\theta}$, hence $[L_x, V(\vec{r})] \neq 0$. Similarly $[L_y, V(\vec{r})] \neq 0$

But $L_z = (-i\hbar)\frac{\partial}{\partial\phi}$. And $V(\vec{r})$ is independent of ϕ - Thus

$$[L_z, V(\vec{r})] = 0$$

But $$[L^2, V(\vec{r})] \neq 0.$$

(d) The ground state (without normalization Const)

$$\phi_{000} = e^{-\alpha^2 r^2 \sin^2\theta/2}\, e^{-\alpha_z^2 r^2 \cos^2\theta}$$

Then $L_z \phi_{000} = 0 = 0 \cdot \phi_{000}$

Thus ϕ_{000} is eigenfunction of L_z with eigenvalue 0.

(e) $$\phi_{100} = \cancel{\cos}\ r\sin\theta\cos\phi\ e^{-\alpha^2 r^2 \sin^2\theta/2} \cdot e^{-\alpha_z^2 r^2 \cos^2\theta/2}$$

$$L_z \phi_{100} = \left(-r\sin\theta\sin\phi\ e^{-\alpha^2 r^2 \sin^2\theta/2}\, e^{-\alpha_z^2 r^2 \cos^2\theta/2}\right)(-i\hbar)$$

$$= -\phi_{010}.(-i\hbar)$$

Simillarly

$$L_z \phi_{010} = \phi_{100}\ (-i\hbar)$$

Let $\psi = \alpha\phi_{100} + \beta\phi_{010}$

$$L_z\psi = \lambda\psi \Rightarrow L_z(\alpha\phi_{100} + \beta\phi_{010}) = \lambda(\alpha\phi_{100} + \beta\phi_{010})$$

$$\Rightarrow (-i\hbar)\left[\alpha(-\phi_{010}) + \beta\phi_{100}\right] = \lambda\alpha\phi_{100} + \lambda\beta\phi_{010}$$

$$\Rightarrow +i\hbar\alpha = \lambda\beta \quad \text{and} \quad (-i\hbar)\beta = \lambda\alpha$$

$$\Rightarrow \frac{\alpha}{\beta} = \frac{\lambda}{i\hbar} \qquad \frac{\alpha}{\beta} = -\frac{i\hbar}{\lambda} \Rightarrow \lambda^2 = \hbar^2$$

$$\Rightarrow \lambda = \pm\hbar$$

if $\lambda = \hbar$ $\quad i\alpha = \beta$

$\lambda = -\hbar$ $\quad -i\alpha = \beta$

$|\alpha|^2 + |\beta|^2 = 1$

$$\Rightarrow \psi_{\pm} = \left(\phi_{100} \pm i\phi_{010}\right)\frac{1}{\sqrt{2}} \qquad \lambda = \hbar$$

$$= \frac{1}{\sqrt{2}}\, r e^{i\phi} \sin\theta\ e^{-r^2\sin^2\theta\,\alpha^2/2}\, e^{-\alpha_z^2 r^2 \cos^2\theta/2}$$

$$\psi_- = \frac{1}{\sqrt{2}}\, r e^{-i\phi} \sin\theta\ e^{-r^2\sin^2\theta\,\alpha^2/2}\, e^{-\alpha_z^2 r^2 \cos^2\theta/2}.$$

You could have done this by inspection.

④ (a) $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n$

Then

$$I = \int_{-1}^{1} P_n(x) P_m(x)\, dx =$$

$$= \frac{1}{2^{m+n} m!\, n!} \int_{-1}^{1} D^{(m)} (x^2-1)^m D^n (x^2-1)^n\, dx \qquad D = \frac{d}{dx}$$

Assume $n > m$

$$= \frac{1}{2^{m+n} m!\, n!} \left[D^m (x^2-1)^m D^{n-1} (x^2-1)^n \Big|_{-1}^{1} - \int_{-1}^{1} D^{(m+1)} (x^2-1)^m D^{n-1} (x^2-1)\, dx \right]$$

First term is zero because $D^{n-1}(x^2-1)^n = (x^2-1)[\cdots]$

or $D^k (x^2-1)^n = (x^2-1)^{n-k} [\cdots] = 0$ at $x = \pm 1$.

$$I = \frac{1}{2^{m+n} m!\, n!} \int_{-1}^{1} D^{(m+n)} (x^2-1)^m (x^2-1)^n\, dx = 0$$

since $n > m$
$m+n > 2m$
$=$ degree of $(x^2-1)^m$.

$$I = \frac{1}{2^{2n} (n!)^2} \int_{-1}^{1} D^{2n} (x^2-1)^n \cdot (x^2-1)^n\, dx$$

when $m = n$.

$D^{2n}(x^{2n} - \underbrace{x^{2n-2} \cdots}_{0})$, $D^{2n} x^{2n} = (2n)!$

$$= \frac{1}{2^{2n} \cdot (n!)^2} \cdot \int_{-1}^{1} (2n)!\, (x^2-1)^n\, dx$$

$$= \frac{1}{2^{2n} (n!)^2} (2n)! \int_{0}^{\pi} \sin^{(2n+1)} \theta\, d\theta$$

$$= \frac{(2n)!}{2^{2n} (n!)^2} \cdot \frac{(2n)(2n-2)\cdots}{(2n+1)(2n-1)\cdots} \cdot 2 = \frac{2}{2n+1}.$$

4 (b) $$G(t,x) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_n P_n(x) t^n$$

Note: $$\frac{dG}{dt} = \frac{(x-t)}{(1-2xt+t^2)^{3/2}} \Rightarrow (1-2xt+t^2)\frac{dG}{dt} = (x-t)G$$

$$\Rightarrow (1-2xt+t^2)\sum_n P_n(x) n t^{n-1} = (x-t)\sum_n P_n(x) t^n$$

$$\Rightarrow \sum_n \underbrace{(n+1) P_{n+1}(x)}\, t^n = \sum_n \underbrace{\left((2n+1)x P_n(x) - n P_{n-1}(x)\right)}\, t^n$$ Adjusting terms

$$\Rightarrow \boxed{(n+1) P_{n+1}(x) = (2n+1)\, x P_n(x) - n P_{n-1}(x)}$$ First identity

$\Rightarrow$ Now, $$\frac{dG}{dx} = \frac{t}{\sqrt{(1-2xt+t^2)^3}} = \frac{t}{(x-t)}\frac{dG}{dt}$$

$$\Rightarrow (x-t)\frac{dG}{dx} = t\frac{dG}{dt}$$

$$\Rightarrow \sum (x-t)\cdot P_n' t^n = \sum t P_n n t^{n-1}$$

$$\Rightarrow \sum (x P_n' - P_{n-1}') t^n = \sum n P_n t^n$$

$$\Rightarrow \boxed{x P_n' - P_{n-1}' = n P_n}$$ Second identity

$\Rightarrow$ Also from the first identity (take a derivative and lower n by 1)

$$n P_n'(x) = (2n-1)\, x P_{n-1}' + (2n-1) P_{n-1} - (n-1) P_{n-2}'$$
$$= (2n-1)\, x P_{n-1}' + (2n-1) P_{n-1} + (n-1)\underbrace{\left[(n-1) P_{n-1} - x P_{n-1}'\right]}_{\text{From second identity}}$$

$$= n x P_{n-1}' + n^2 P_{n-1}$$

$$\Rightarrow \boxed{P_n' - x P_{n-1}' = n P_{n-1}}$$ Third identity.

$$\Rightarrow \boxed{(x^2-1) P_n' = n x P_n - n P_{n-1}}$$ Required Identity

obtained by multiplying Second Id by x and subtracting from 3rd Identity.

Q5. (a) $T_a = \exp[-ia\hat{P}_x/\hbar]$ $\qquad P_x = -i\hbar\frac{d}{dx}$

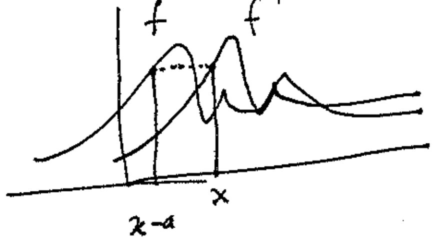
$$= \exp\left[-a\frac{d}{dx}\right]$$

$$\therefore\ T_a f(x) = \exp\left[-a\frac{d}{dx}\right] f(x)$$

$$= \left[\mathbb{1} - \frac{a}{1}\frac{d}{dx} + \frac{a^2}{2}\frac{d^2}{dx^2} - \cdots\right] f(x)$$

$$= \sum_n f^{(n)}(x)\cdot\frac{a^n}{n!}(-1)^n$$

$$\therefore\ f'(x) = f(x-a)$$



(b) Clearly,

$$\left[T_a, \frac{P_x^2}{2m}\right] = 0$$

Now $T_a\left(V(x)f(x)\right) = V(x+a)\cdot f(x+a)$

$$= V(x+a)\ T_a f(x)$$

$$\Rightarrow\quad T_a V(x) = V(x+a)\ T_a$$

Thus if $V(x+a) = V(a) \Rightarrow T_a V(x) = V(x)\ T_a$

$$\Rightarrow [T_a, V(x)] = 0$$

$$\Rightarrow [T_a, \hat{H}] = 0$$

(c) Exactly same as (a).