1. Prove the following identities:

   (a) $[A, B + C] = [A, B] + [A, C]$.

   (b) $[A, BC] = [A, B]C + B[A, C]$.

   (c) $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$.

   (d) $\left[f\left(\hat{X}\right), \hat{P}\right] = i\hbar\frac{df}{dx}$, where f is operator function of $\hat{X}$.

2. For any operator A, show that $\left(A + A^\dagger\right)$, $i\left(A - A^\dagger\right)$ and $AA^\dagger$ are hermitian operators.

3. Prove:
$$e^{+A}Be^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \cdots$$
   and if A and B commute with their commutator then
$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]}.$$

4. Let $\mathcal{H}$ be a Hilbert space. Prove Schwarz inequality, that is, for any two vectors, $f, g \in \mathcal{H}$, show that
$$|\langle f, g\rangle|^2 \leq \langle f, f\rangle \langle g, g\rangle .$$
   [Hint: Consider $\langle f + \lambda g, f + \lambda g\rangle \geq 0$. Now find λ such that the lhs is minimum.]

5. Let a time dependent observable be represented by a hermitian operator$\hat{\Omega}(t)$. If the system is in state $\Psi(t)$, show that
$$\frac{d}{dt}\left\langle\hat{\Omega}\right\rangle = \frac{i}{\hbar}\left\langle\left[\hat{H}, \hat{\Omega}\right]\right\rangle + \left\langle\frac{\partial\hat{\Omega}}{\partial t}\right\rangle$$
   where $\hat{H}$ is the hamiltonian operator. [Hint: If $f(t)$ and $g(t)$ are any two states, then prove that
$$\frac{d}{dt}\langle f(t), g(t)\rangle = \left\langle\frac{d}{dt}f(t), g(t)\right\rangle + \left\langle f(t), \frac{d}{dt}g(t)\right\rangle$$
   using the properties of inner product and
$$\frac{d}{dt}\langle f(t), g(t)\rangle = \lim_{\Delta t\to 0}\frac{\langle f(t+\Delta t), g(t+\Delta t)\rangle - \langle f(t), g(t)\rangle}{\Delta t}.$$

6. If the hamiltonian operator of a system is given by $\hat{H} = \hat{P}^2/2m + V(\hat{X})$, then prove the Ehrenfest's theorem:
$$\begin{aligned}\frac{d}{dt}\left\langle\hat{X}\right\rangle &= \frac{1}{m}\left\langle\hat{P}\right\rangle \\ \frac{d}{dt}\left\langle\hat{P}\right\rangle &= \left\langle -V'(\hat{X})\right\rangle.\end{aligned}$$
   [illegible]

7. Consider a quantum system consisting of a particle in a conservative force field. The energy spectrum is $\{E_1, E_2, \ldots\}$ with corresponding normalized stationary states $\{\phi_1, \phi_2, \ldots\}$. Let x_0 be an eigenvalue of the position operator with eigenvector ξ_{x_0}. Let $\alpha_1 = |\langle\xi_{x_0}, \phi_1\rangle|^2$ and $\alpha_2 = |\langle\xi_{x_0}, \phi_2\rangle|^2$. Let $\Psi(t)$ denote the state of the system at time t. Express, in terms of α_1, α_2 and eigenenergies, the answers to the following questions:

(a) If $\Psi(0) = \phi_1$, what is the probability density of finding the particle at x_0 at time t?

(b) If $\Psi(0) = \phi_2$, what is the probability density of finding the particle at x_0 at time t?

(c) If $\Psi(0) = (\phi_1 + \phi_2)/\sqrt{2}$, what is the probability density of finding the particle at x_0 at time t? What is the maximum probability density? And minimum?

8. Here is an example of time dependent Hamiltonian: An electron in an oscillating electric field is described by a Hamiltonian operator

$$\hat{H} = \frac{\hat{P}^2}{2m} - (eE_0 \cos \omega t)\, x.$$

where E_0 is the amplitude of the electric field. Calculate $d\langle \hat{X} \rangle / dt$, and $d\langle \hat{P} \rangle / dt$.

9. The potential energy of a harmonic oscillator is given by $V(x) = m\omega^2 x^2/2$. Assuming that $\langle \hat{X} \rangle = \langle \hat{P} \rangle = 0$, find the lower limit to the expectation value of the Hamiltonian operator. [Hint: Use the uncertainty principle.]

10. The first excited state of the harmonic oscillator is given by

$$\psi_1(x) = \left(\frac{2\alpha}{\sqrt{\pi}}\right)^{1/2} (\alpha x)\, e^{-\alpha^2 x^2/2}.$$

Find ΔX and ΔP and check uncertainty principle. $\left(\cancel{\alpha \neq \frac{\hbar}{m\omega}}\right)$ $\left(\alpha^2 = \frac{m\omega}{\hbar}\right)$

Tutorial 3.

Q1. (a) $[A, B+C] = A(B+C) - (B+C)A = (AB-BA) + (AC-CA)$
$$= [A,B] + [A,C].$$

(b) $[A, BC] = ABC - BCA = ABC - BAC + BAC - BCA$
$$= [A,B]\,C + B[A,C].$$

(c) Use brute force.

(d) $[f(\hat{x}), \hat{p}]\,g(x) = [f(\hat{x})\hat{p} - \hat{p} f(\hat{x})]\,g(x)$

$$= f(x)\cdot\left(-i\hbar\frac{d}{dx}\right)g(x) - \left(-i\hbar\frac{d}{dx}\right) f(x)g(x)$$

$$= f(x)\,\hat{p}\,g(x) - f(x)\,\hat{p}\,g(x) - \left(-i\hbar\frac{df}{dx}\right)g(x)$$

$$= i\hbar\frac{df}{dx}(x)\,g(x)$$

$$\Rightarrow \quad [f(\hat{x}), \hat{p}] = i\hbar\frac{df}{dx}$$

Q2. $(A + A^\dagger)^\dagger = A^\dagger + (A^\dagger)^\dagger = A^\dagger + A.$

$[i(A - A^\dagger)]^\dagger = (A^\dagger - (A^\dagger)^\dagger)\, i^\dagger = (-i)(A^\dagger - A) = i(A - A^\dagger)$

$(AA^\dagger)^\dagger = A^\dagger (A^\dagger)^\dagger = A^\dagger A.$

Q3. Step 1: Prove that $\frac{d}{d\lambda} e^{\lambda A} = A e^{\lambda A}$ (remember since A is an operator this identity is not obvious.)

using
$$\frac{d}{d\lambda} e^{\lambda A} = \lim_{\Delta\lambda \to 0} \frac{e^{(\lambda+\Delta\lambda)A} - e^{\lambda A}}{\Delta\lambda}$$

Step 2. Consider

$$f(\lambda) = e^{\lambda A} B e^{-\lambda A}$$

then $f(\lambda) = f(0) + \lambda f'(0) + \frac{\lambda^2}{2!} f''(0) + \cdots \qquad ①$

Now $f(0) = B, \quad f'(0) = \frac{d}{d\lambda}\left(e^{\lambda A} B e^{-\lambda A}\right)\Big|_{\lambda=0}$

$$= \left(A e^{\lambda A} B e^{-\lambda A} + e^{\lambda A} B (-A e^{-\lambda A})\right)\Big|_{\lambda=0}$$

$$= e^{\lambda A} [A,B] e^{-\lambda A}\Big|_{\lambda=0}$$

$$= [A,B].$$

and $f''(0) = [A, [A,B]]$ etc.

Now, substitute in ① and put $\lambda = 1$.

Q4. Now $\langle f+\lambda g, f+\lambda g\rangle \geq 0$ <u>true for all λ</u>

$$\Rightarrow \underbrace{|f|^2 + |\lambda|^2 |g|^2 + \lambda \langle f,g\rangle + \lambda^* \langle g,f\rangle}_{\text{lhs}} \geq 0$$

Now lhs will be minimum when

$$\lambda = -\frac{\langle g, f\rangle}{\cancel{4}|g|^2}$$

(while finding min, remember λ is complex No.)

Thus

$$\Rightarrow |f|^2 + \frac{|\langle f,g\rangle|^2}{\cancel{4}|g|^4}|g|^2 - \frac{|\langle f,g\rangle|^2}{\cancel{4}|g|^2} - \frac{|\langle f,g\rangle|^2}{|g|^2} \geq 0$$

$$\Rightarrow \underline{|f|^2|g|^2 \geq |\langle f,g\rangle|^2}$$

Q5. Step 1. Now

$$\frac{d}{dt}\langle f(t), g(t)\rangle = \lim_{\Delta t\to 0} \frac{\langle f(t+\Delta t), g(t+\Delta t)\rangle - \langle f(t), g(t)\rangle}{\Delta t}$$

$$= \lim_{\Delta t\to 0}\left\{ \frac{\langle f(t+\Delta t), g(t+\Delta t)\rangle - \langle f(t), g(t+\Delta t)\rangle}{\Delta t} + \frac{\langle f(t), g(t+\Delta t)\rangle - \langle f(t), g(t)\rangle}{\Delta t}\right\}$$

$$= \lim_{\Delta t\to 0}\left[\left\langle \frac{f(t+\Delta t)-f(t)}{\Delta t}, g(t+\Delta t)\right\rangle + \left\langle f(t), \frac{g(t+\Delta t)-g(t)}{\Delta t}\right\rangle\right]$$

$$= \left\langle \frac{df}{dt}, g\right\rangle + \left\langle f, \frac{dg}{dt}\right\rangle.$$

Step 2.

$$\frac{d}{dt}\left(\hat{\Omega}(t)\right) = \frac{d}{dt}\langle \Psi(t)\, \hat{\Omega}(t)\, \Psi(t)\rangle$$

$$= \left\langle \frac{d\Psi}{dt}, \hat{\Omega}(t)\Psi(t)\right\rangle + \left\langle \Psi(t), \left(\frac{d\hat{\Omega}}{dt}(t)\right)\Psi(t)\right\rangle + \left\langle \Psi, \hat{\Omega}\left(\frac{d\Psi}{dt}\right)\right\rangle$$

By schrödinger eq. $i\hbar \frac{d\Psi}{dt} = \hat{H}\Psi$

Then $$\frac{d}{dt}\langle\hat{\Omega}(t)\rangle = \left\langle \frac{1}{i\hbar}\hat{H}\psi, \hat{\Omega}\psi\right\rangle + \left\langle \Psi, \hat{\Omega}\left(\frac{1}{i\hbar}\hat{H}\psi\right)\right\rangle + \left\langle \Psi, \left(\frac{\partial\hat{\Omega}(t)}{\partial t}\right)\Psi\right\rangle$$

$$= i\hbar^{-1}\langle \hat{H}\hat{\Omega} - \hat{\Omega}\hat{H}\rangle + \left\langle \frac{\partial\hat{\Omega}(t)}{\partial t}\right\rangle$$

$$= i\hbar^{-1}\langle [H, \hat{\Omega}(t)]\rangle + \left\langle \frac{\partial\hat{\Omega}(t)}{\partial t}\right\rangle.$$

Q6. Using the result of 5

$$[\hat{H}, \hat{X}] = \frac{1}{2m}[P^2, \hat{X}] = \frac{1}{2m}\left\{\hat{P}[\hat{P}, \hat{X}] + [\hat{P}, \hat{X}]\hat{P}\right\}$$

$$= \frac{1}{2m}\left(2\hat{P}(i\hbar)\right) = -i\hbar\frac{\hat{P}}{m}$$

$$[\hat{H}, \hat{P}] = [V(\hat{X}), \hat{P}] = i\hbar\frac{dV}{dx}$$

Then $$\frac{d}{dt}\langle\hat{X}\rangle = \frac{i}{\hbar}\langle[\hat{H}, \hat{X}]\rangle = \frac{i}{\hbar}(-i\hbar)\frac{\langle\hat{P}\rangle}{m} = \frac{\langle\hat{P}\rangle}{m}$$

$$\frac{d}{dt}(\hat{P}) = \frac{i}{\hbar}\langle[\hat{H}, \hat{P}]\rangle = \frac{i}{\hbar}(i\hbar)\left\langle\frac{dV}{dx}\right\rangle.$$

$$= \langle -V'(x)\rangle.$$

Q7. (a) if $\Psi(0) = \phi_1$ then $\Psi(t) = e^{-iE_1t/\hbar}\phi_1$

$$P_{\hat{X}}(x_0; t) = |\langle \xi_{x_0}, \Psi(t)\rangle|^2 = |\langle \xi_{x_0}, \phi_1\rangle|^2 = \alpha_1^2$$

(b) Similarly

$$P_{\hat{X}}(x_0, t) = \alpha_2^2$$

(c) $$\psi(0) = \frac{1}{\sqrt{2}}(\phi_1 + \phi_2)$$

Then $$\Psi(t) = \frac{1}{\sqrt{2}}\left[\phi_1 e^{-iE_1t/\hbar} + \phi_2 e^{-iE_2t/\hbar}\right]$$

Thus

$$\rho_{\hat{x}}(x_0,t) = \left|\left\langle \xi_{x_0}, \frac{1}{\sqrt{2}}\phi_1 e^{-iE_1t/\hbar}\right\rangle + \left\langle \xi_{x_0}, \frac{1}{\sqrt{2}}\phi_2 e^{-iE_2t/\hbar}\right\rangle\right|^2$$

$$= \frac{1}{2}\alpha_1^2 + \frac{1}{2}\alpha_2^2 + \cancel{2\,\mathrm{Re}}\Big[\frac{1}{2}\langle \xi_{x_0},\phi_1\rangle^* \langle \xi_{x_0},\phi_2\rangle e^{-i(E_2-E_1)t/\hbar} + \frac{1}{2}\langle \xi_{x_0},\phi_1\rangle \langle \xi_{x_0},\phi_2\rangle^* e^{-i(E_1-E_2)t/\hbar}$$

$$= \frac{1}{2}\alpha_1^2 + \frac{1}{2}\alpha_2^2 + \frac{1}{2}\cdot 2\cdot \mathrm{Re}\left(\alpha_1\alpha_2^* e^{-i(E_1-E_2)t/\hbar}\right)$$

Max when $e^{-i(E_1+E_2)t/\hbar} = +1$

$$= \frac{1}{2}|\alpha_1+\alpha_2|^2$$

Min when $e^{-i(E_1-E_2)t/\hbar} = -1$

$$= \frac{1}{2}|\alpha_1-\alpha_2|^2$$

Q8.

$$\frac{d\langle\hat{x}\rangle}{dt} = \frac{\langle\hat{p}\rangle}{m}$$

$$\frac{d\langle\hat{p}\rangle}{dt} = +eE_0\cos\omega t$$

Q9.

$$\langle\hat{H}\rangle = \frac{\langle p^2\rangle}{2m} + \frac{1}{2}m\omega^2\langle x^2\rangle$$

10. Given $\psi_1(x) = \left(\frac{2\alpha}{\sqrt{\pi}}\right)^{1/2} (\alpha x)\, e^{-\alpha^2 x^2/2}$

(a) $\langle x \rangle = 0 \qquad \langle p \rangle = 0$

(b) $$\langle x^2 \rangle = \left(\frac{2\alpha}{\sqrt{\pi}}\right) \alpha \int_{-\infty}^{\infty} (\alpha x)^2 e^{-\alpha^2 x^2} x^2\, dx$$

$$= \frac{3}{2} \Big/ a^2$$

(c) $\langle p^2 \rangle = \frac{3a^2\hbar^2}{2}$

(d) $\Delta x = \sqrt{\frac{3}{2}} \frac{1}{a} \qquad \Delta P = \sqrt{\frac{3}{2}}\, a\hbar$

$$\boxed{\Delta x \Delta P = \frac{3}{2}\hbar > \frac{1}{2}\hbar}$$