1. Which of the following sets are vector spaces? (Assume usual function addition. Check only closure and existence of inverse.)

   (a) Piecewise continuous functions on $[a, b]$.
   (b) Twice differentiable functions on $[a, b]$.
   (c) Functions on $[0, a]$ satisfying the boundary conditions $f(0) = f(a)$.
   (d) Functions on $[0, a]$ satisfying the boundary conditions $f(0) = 0$ and $f(a) = 2$.
   (e) Functions satisfying the differential equation $y'' + y^2 = 0$.
   (f) Functions satisfying the differential equation $y'' + y = 0$.

2. Let $f_n : [0, \pi] \to \mathbb{R}$ such that $f_n(x) = \sin(nx)$ for $n = 1, 2, \ldots$. Show that the set $\{f_n | n = 1, 2, \ldots\}$ is orthogonal with respect to the inner product

$$\langle f_n; f_m \rangle = \int_0^\pi f_n(x) f_m(x) dx.$$

   Normalize these functions.

3. Prove Schwarz inequality,

$$\left| \int_a^b f^*(x) g(x) dx \right|^2 \leq \left[\int_a^b |f(x)|^2 \, dx \right] \left[\int_a^b |g(x)|^2 \, dx \right]$$

   for $f, g \in L_2([a, b])$. Use this identity to show that $L_2([a, b])$ is a vector space.

4. For what range of ν, is the function $f(x) = x^\nu$ in $L_2([0, 1])$. Assume ν to be real but not necessarily positive. For a specific case of $\nu = 1/2$, is f in $L_2([0, 1])$? What about $xf(x)$? And $(d/dx)f$?

5. Prove the following:

   (a) $(cA)^\dagger = c^* A^\dagger$
   (b) $(A + B)^\dagger = A^\dagger + B^\dagger$. Thus the sum of two hermitian operators is hermitian.
   (c) Show that $(AB)^\dagger = B^\dagger A^\dagger$. Thus the product of two hermitian operators is hermitian if they commute.
   (d) Hamiltonian operator

$$-\frac{\hbar^2}{2m} \hat{D}^2 + V(\hat{X})$$

   is hermitian. Here $V(\hat{X})$ is a function of the operator $\hat{X}$ and

$$\left(V(\hat{X}) f \right)(x) = V(x) f(x).$$

   Assume that the function $V(x)$ is real valued.

6. Let V be a finite dimensional inner product space. Let M_A be the matrix of an operator A with respect to an orthonormal basis. Show that

$$M_{A^\dagger} = [M_A^*]^T.$$

7. Show that the eigenvalues of hermitian operator are real. Also show that the eigenfunctions corresponding to distinct eigenvalues are orthogonal.

8. Let $W = \{f(\phi) \in L_2([0, 2\pi]) \mid f(0) = f(2\pi) \text{ and } f'(0) = f'(2\pi)\}$. Consider an operater $\hat{Q} = d^2/d\phi^2$ on W. Is $\hat{Q}$ hermitian? Find its eigenfunctions and eigenvalues.

9. The position operator $\hat{X} : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ is defined as

$$\left(\hat{X} f\right)(x) = x f(x).$$

Find the eigenvalues and eigenfunctions of the position operator.

10. The matrix of an operator A on $\mathbb{R}^3$ is given by

$$\begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ a & 0 & a \end{bmatrix}.$$

Find the eigenvalues and eigenvectors.

Tutorial 2

Q1. (a) Yes, infinite dimensional.

(b) Yes, infinite dimensional

(c) Yes, infinite dimencional

(d) No. $(f+g)(a) = 4$!

(e) No.

(f) Yes, Two dimensional.

Q2. It is easy to see that

$$\langle f_n, f_m \rangle = \int_0^{\pi} \sin(nx)\sin(mx) = \frac{\pi}{2}\delta_{m,n}$$

Thus the set $\{f_n\}$ is orthogonal. Each f_n can be normalized by multiplying by $\sqrt{2/\pi}$.

Q3. Let α and β be any two ~~complex~~ real numbers then

Then $(\alpha-\beta)^2 \geq 0 \Rightarrow \frac{1}{2}(\alpha^2+\beta^2) \geq \alpha\beta$.

Now

$$\frac{1}{|f||g|}\left|\int_a^b f^*(x)g(x)dx\right|$$

$$\leq \frac{1}{|f|\cdot|g|}\int_a^b |f(x)|\cdot|g(x)|\,dx$$

$$\leq \int_a^b \frac{1}{2}\left(\frac{|f(x)|^2}{|f|^2} + \frac{|g(x)|^2}{|g|^2}\right)dx$$

put $\alpha = \dfrac{|f(x)|}{|f|}$

$\beta = \dfrac{|g(x)|}{|g|}$

$$= \frac{1}{2}\left\{\frac{1}{|f|^2}\int_a^b |f(x)|^2 dx + \frac{1}{|g|^2}\int_a^b |g(x)|^2 dx\right\}$$

$$= 1.$$

證畢.

$$|\alpha - \lambda\beta|^2 = \langle\alpha,\alpha\rangle + \lambda^2\langle\beta,\beta\rangle - \lambda\langle\alpha,\beta\rangle - \lambda\langle\beta,\alpha\rangle \geq 0$$

$$= \langle\alpha,\alpha\rangle + \lambda^2\langle\beta,\beta\rangle \geq \lambda\langle\alpha,\beta\rangle + \langle\beta,\alpha\rangle$$

$$\lambda = \frac{\langle\alpha,\beta\rangle}{\langle\beta,\beta\rangle}$$

$$\langle\alpha,\alpha\rangle + \frac{\langle\alpha,\beta\rangle^2}{\langle\beta,\beta\rangle} \geq \frac{\langle\alpha,\beta\rangle^2}{\langle\beta,\beta\rangle} + \langle\beta,\alpha\rangle$$

$$|\alpha - \lambda\beta|^2 = \langle\alpha,\alpha\rangle + \lambda^2\langle\beta,\beta\rangle \cancel{+} \lambda\langle\alpha,\beta\rangle$$
$$- \lambda\langle\beta,\alpha\rangle \geq 0$$

$$\Rightarrow \langle\alpha,\alpha\rangle + \lambda^2\langle\beta,\beta\rangle \geq \lambda\langle\alpha,\beta\rangle + \langle\beta,\alpha\rangle$$

$$\lambda = \frac{\langle \alpha,\beta \rangle}{\langle\beta,\beta\rangle}$$

$$\langle\alpha,\alpha\rangle \cancel{\langle\beta,\beta\rangle} + \frac{\langle\alpha,\beta\rangle^2}{\langle\beta,\beta\rangle} \geq \frac{\langle\alpha,\beta\rangle^2}{\langle\beta,\beta\rangle} + \langle\beta,\alpha\rangle$$

Q4. If $(2\nu+1) > 1$, or $\nu > -\frac{1}{2}$

$$\int_0^1 x^{2\nu} dx = \left.\frac{x^{2\nu+1}}{2\nu+1}\right|_0^1 = \frac{1}{2\nu+1}$$

If $2\nu+1 = 0$ or $\nu = -\frac{1}{2}$,

$$\int_0^1 \frac{1}{x} dx = \ln x \Big|_0^1 = -\infty$$

If $2\nu+1 < 0$, let $2\nu+1 = -\mu$

$$\int_0^1 x^{2\nu} dx = \left.\frac{x^{-\mu}}{(-\mu)}\right|_0^1 = +\infty$$

Thus for $\nu \in (-\frac{1}{2}, \infty)$, $f(x) = x^{\nu}$ is square integrable and is in $L_2([0,1])$.

If $\nu = \frac{1}{2}$, that is $f(x) = \sqrt{x}$, clearly is in $L_2([0,1])$. So if $x f(x) = x^{3/2}$. However $\frac{df}{dx} = \frac{1}{2\sqrt{x}}$ is not in $L_2([0,1])$.

Q5. (a) $\langle u, (cA)v \rangle = c \langle u, Av \rangle$

$$\langle c^* A^{\dagger} u, v \rangle = (c^*)^* \langle A^{\dagger} u, v \rangle = c^* \langle u, Av \rangle$$

Thus: $(cA)^{\dagger} = c^* A^{\dagger}$

(b) $\langle u, (A+B)v \rangle = \langle u, Av \rangle + \langle u, Bv \rangle$

$$= \langle A^{\dagger}u, v \rangle + \langle B^{\dagger}u, v \rangle$$

$$= \langle A^{\dagger}u + B^{\dagger}u, v \rangle = \langle (A^{\dagger}+B^{\dagger})u, v \rangle$$

thus $(A+B)^{\dagger} = A^{\dagger} + B^{\dagger}$.

(c) $\langle u, ABv \rangle = \langle A^{\dagger}u, Bv \rangle = \langle B^{\dagger}A^{\dagger}u, v \rangle$

Thus $(AB)^{\dagger} = B^{\dagger}A^{\dagger}$.

(d) ~~[illegible]~~. Thus if A and B are hermitian

$\Rightarrow$ $(AB)^{\dagger} = BA$. AB is hermitian if $(AB)^{\dagger} = AB$.

i.e. when $BA = AB$ or $[A, B] = 0$.

Q5(d.) ~~[illegible]~~ $B = \hat{D}^2$ $\quad B^{+} = \hat{D}^{+}\hat{D}^{+} = (-\hat{D})(-\hat{D}) = \hat{D}^2 = B$

And it is obvious that $V(\hat{x})$ is hermitian. Thus. sum

$$-\frac{\hbar^2}{2m}\hat{D}^2 + V(\hat{x}) \text{ is also hermitian.}$$

Q6. In orthonormal Basis, the matrix of ~~an~~ operator is given by

$$A_{ij} = \langle e_i, A e_j \rangle$$

$$[A^{+}]_{ij} = \langle e_i, A^{+} e_j \rangle$$

$$= \langle A e_i, e_j \rangle = \overline{\langle e_j, A e_i \rangle} = \overline{A_{ji}}$$

$$= [A^{*}]_{ji}$$

$$= [[A^{*}]^{T}]_{ij}$$

Thus $\quad A^{+} = [A^{*}]^{T}$.

Q7. For hermitian operator A, let λ be eigenvalue with evector x_λ, That is

$$A x_\lambda = \lambda x_\lambda$$

Consider $\langle A x_\lambda, x_\lambda \rangle = \lambda^{*} \langle x_\lambda, x_\lambda \rangle$

and $\langle x_\lambda, A x_\lambda \rangle = \lambda \langle x_\lambda, x_\lambda \rangle$

Subtract: $\quad 0 = (\lambda^{*} - \lambda) \langle x_\lambda, x_\lambda \rangle$

$\Rightarrow \quad \lambda^{*} = \lambda \quad \Rightarrow \quad \lambda$ must be real.

Let $\quad A x_{\lambda_1} = \lambda_1 x_{\lambda_1}$

$\quad A x_{\lambda_2} = \lambda_2 x_{\lambda_2}$

Thus $\quad \langle x_{\lambda_2}, A x_{\lambda_1} \rangle = \lambda_1 \langle x_{\lambda_2}, x_{\lambda_1} \rangle$

and. $\quad \langle A x_{\lambda_2}, x_{\lambda_1} \rangle = \lambda_2 \langle x_{\lambda_2}, x_{\lambda_1} \rangle$

Subtract: $\quad 0 = (\lambda_1 - \lambda_2) \langle x_{\lambda_2}, x_{\lambda_1} \rangle$

if $\quad \lambda_1 \neq \lambda_2$ then $\quad \langle x_{\lambda_1} | x_{\lambda_2} \rangle = 0$

Q8. Since

$$\langle \frac{d^2}{d\phi^2} f, g\rangle = \int_0^{2\pi} \frac{d^2 f}{d\phi^2}^*(\phi)\, g(\phi)\, d\phi$$

$$= \int_0^{2\pi} f^*(\phi) \frac{d^2}{d\phi^2} g(\phi)\, d\phi \qquad \text{Integrate by parts twice.}$$

$$= \langle f, \frac{d^2}{d\phi^2} g\rangle$$

Thus $d^2/d\phi^2$ is hermitian.

Now

$$\frac{d^2 u_\lambda(\phi)}{d\phi^2} = \lambda u_\lambda(\phi)$$

$$\Rightarrow \quad u_\lambda^{(1)} = A e^{-\sqrt{\lambda}\,\phi} \quad \text{and} \quad u_\lambda^{(2)} = B e^{\sqrt{\lambda}\,\phi} \qquad \lambda \neq 0$$

Since $u_\lambda^{(1)}(0) = u_\lambda^{(1)}(2\pi) \Rightarrow \pm 2\pi\sqrt{\lambda} = \pm 2\pi i n$

$$\Rightarrow \boxed{\sqrt{\lambda} = in}$$

Thus eigenvalues are $-n^2$ with two eigenvectors

$e^{in\lambda}$ and $e^{-in\lambda}$ for $n = 1, 2, \ldots$

and ~~for 0 eigenvalue there is no ev~~

For 0 eigenvalue two sol^n are

1 and ϕ (not in W): hence only one eigenvalue.

Q9. $(\hat{X} f)(x) = x f_\lambda(x) = \lambda f_\lambda(x)$

Now let $f_\lambda(x) = \delta(x-\lambda)$

Clearly $x f_\lambda(x) = \lambda f_\lambda(x)$

Alt. check $\displaystyle\int_{-\infty}^{\infty} (x-\lambda)^2 \delta(x-\lambda)\delta(x-\lambda)\, dx = (\lambda-\lambda)^2 = 0.$

For every $\lambda \in \mathbb{R}$ is an eigenvalue of $\hat{X}$ operator with eigenfunction $\delta(x-\lambda)$.

10.

$$\det \begin{bmatrix} a-\lambda & 0 & b \\ 0 & c-\lambda & 0 \\ b & 0 & a-\lambda \end{bmatrix} = (a-\lambda)\left[(c-\lambda)(a-\lambda) - 0\right] + b\left[0 - b(c-\lambda)\right] = 0$$

$$\Rightarrow (c-\lambda)\left[(a-\lambda)^2 - b^2\right] = 0$$

$\Rightarrow$ ~~cλ~~ or

$\Rightarrow \lambda = c$ or $a - \lambda = \pm b$

$\Rightarrow$ The evs are $a+b$, $a-b$, and c.

(a) Let $\lambda = a+b$ then

$$\begin{bmatrix} -b & 0 & b \\ 0 & c-(a+b) & 0 \\ b & 0 & -b \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$\Rightarrow$ $x = +z$ and $y = 0$

$\Rightarrow$ e vector is $\begin{bmatrix} 1 \\ 0 \\ +1 \end{bmatrix} \xrightarrow{\text{normalize}} \begin{bmatrix} 1 \\ 0 \\ +1 \end{bmatrix} 1/\sqrt{2}$

(b) Similarly show

ev $(a-b)$ has eigenvector $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} 1/\sqrt{2}$

and ev c has eigen vector $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.