1. Two fair and identical dice are thrown.
   (a) Write down the sample space.
   (b) What is the probability that the sum ≤ 4?
   (c) What is the probability that the sum is divisible by three?
   (d) If one forms a two digit number from the outcome, what is the probability that the two digit number is greater than 33?

Gr. 1.12

2. Consider a semicirle $S = \{(x,y) \,|\, x^2 + y^2 = 1,\, y \geq 0\}$. A particle moves back and forth on S with uniform speed. An experiment is performed to find the x coordinate of the particle at some random time. Denote the outcome by X. What is the probability density function $\rho(x)$ for X. Plot $\rho(x)$. Find $\langle x \rangle$, $\langle x^2 \rangle$. Find average value of y coordinate.

3. It can be shown that for an ideal (classical) gas, the probability density function of the distance that molecule travels between collisions be x, is $e^{-x/\lambda}$, where λ is a constant. Show that the average distance between collisions (called the *mean free path*) is λ. Find the probability that the free path is greater than 2λ.

4. Find the fourier series of
$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ 1 & 0 < x < \pi/2 \\ 0 & \pi/2 < x < \pi. \end{cases}$$

5. Find the fourier transform of
$$f(x) = \begin{cases} \cos x & |x| < \pi/2 \\ 0 & |x| > \pi/2 \end{cases}$$

BJ 2.3

6. The family of functions $\delta_L(x)$ is defined by
$$\delta_L(x) = \frac{1}{2\pi} \int_{-L}^{L} e^{ikx} dx.$$
Evaluate the integral and show that $\delta_L(x)$ behaves like a delta function as $L \to \infty$.

Gr. 1.3

7. The gaussian function given by
$$g(x) = A \exp\left[-\frac{(x-a)^2}{2\sigma^2}\right]$$
where a and σ are positive constants.
   (a) Normalize g such that $\int_{-\infty}^{\infty} g(x)dx = 1$.
   (b) Find $\langle x \rangle$, $\langle x^2 \rangle$ and standard deviation of x.
   (c) Sketch the function.

Gr. 1.17

8. The wave function of a particle at $t = 0$ is given by
$$\psi(x,0) = \begin{cases} A\left(a^2 - x^2\right), & |x| \leq a \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find normalization constant A.

(b) Find $\langle x \rangle$, $\langle x^2 \rangle$ and standard deviation σ_x of x.

(c) Find $\langle p \rangle$, $\langle p^2 \rangle$ and standard deviation σ_p of p.

(d) Is your answer consistent with uncertainty principle?

Gr. 2.21
BJ 2.1

9. A free particle has the initial wave function

$$\psi(x,0) = B \exp\left[i\frac{p_0 x}{\hbar}\right] \exp\left[-\frac{|x|}{2\Delta_x}\right].$$

Normalize $\psi(x,0)$. Find momentum space wave function $A(p)$. Suggest a reasonable definition of uncertainty in p and denote it by Δ_p. Show that $\Delta_x \Delta_p \gtrsim \hbar$.

Gr. 2.22

10. Suppose that the momentum space wave function of a gaussian wave packet is given by

$$A(p) = \left(\frac{1}{\sqrt{\pi}\Delta_p}\right)^{\frac{1}{2}} \exp\left[-\frac{(p-p_0)^2}{2\Delta_p^2}\right].$$

Find $\psi(x,t)$ for $t > 0$. What is the width (defined as the standard deviation of x) of the wave packet at time t?

(a) $\Omega = \{(i,j) \mid i,j \in \{1,2,\ldots,6\}\}$

(b) Event A = Sum ≤ 4 or $i+j \leq 4$

$= \{(1,3),(2,2),(3,1)\}$

$P_A = O(A)/O(\Omega) = 1/12$

(c) Event B = Sum is divisible by 3.

$= \{(1,2),(2,1),(1,5),(2,4),(3,3),(4,2),(5,1),(3,6),(4,5),(5,4),$
$(4,5),(6,3),(6,6)\} \Rightarrow O(B) = 13$

$P_B = 13/36$.

(D) Event C :

34 (7)	41 (2)	51 (2)	61 (2)
35 (7)	42 (2)	52 (2)	62 (2)
36 (7)	43 (7)	53 (7)	63 (1)
	44 (1)	54 (1)	64 (1)
	45 (7)	55 (1)	65 (1)
	46 (7)	56 (1)	66 (1)
~~6~~ 3	8	8	8

$P_C = 3/4$

: One of the two is greater than ~~6~~ 3.

: $P(D) + P(E) - P(D \cap E)$

$\frac{1}{2} + \frac{1}{2} - \frac{1}{4} = \frac{3}{4}$ ✓

Q2. Put $x = -\cos\theta \quad y = \sin\theta \qquad \theta \in [0 \to \pi]$

$P(\theta) = \frac{1}{\pi} \Rightarrow$ uniform.

$dx = +\sin\theta \, d\theta$

$d\theta = \frac{dx}{\sqrt{1-x^2}}$

$x = -\cos\theta$

$\langle \cos\theta \rangle = \frac{1}{\pi}\int_0^\pi \cos\theta \, d\theta = 0$

$\langle \cos^2\theta \rangle = \frac{1}{\pi}\int_0^\pi \cos^2\theta \, d\theta = \frac{1}{2}$

$\langle \sin\theta \rangle = \frac{1}{\pi}\int_0^\pi \sin\theta \, d\theta = -\frac{1}{\pi}\left[\cos\theta\right]_0^\pi = -\frac{2}{\pi}$

$P(x)\,dx = P(\theta)\,d\theta$

$= \frac{1}{\pi}\frac{dx}{\sqrt{1-x^2}}$

$\Rightarrow P(x) \equiv \frac{1}{\pi}\frac{1}{\sqrt{1-x^2}}$

$\Rightarrow \langle x \rangle = \int_{-1}^{1} x P(x)\,dx = 0$

$\langle x^2 \rangle = \int_{-1}^{1} x^2 P(x)\,dx$

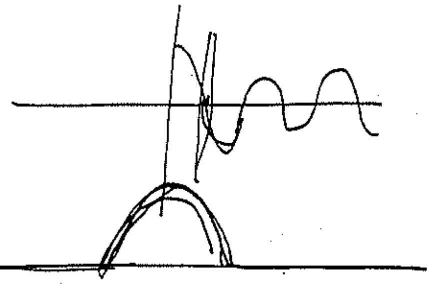
$= \frac{1}{\pi}\int_{-1}^{1} \frac{x^2}{\sqrt{1-x^2}}\,dx$

$x = \cos t \Rightarrow dx = -\sin t \, dt$

$= \frac{1}{\pi}\int_0^\pi \frac{\cos^2 t \, \sin t}{\sin^2 t}\,dt$

$= \frac{1}{\pi}\frac{\pi}{2} = \frac{1}{2}$. ✓

Q5



$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\pi/2}^{\pi/2} \cos x \, e^{-i\omega x} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\pi/2}^{\pi/2} \frac{1}{2} \left[e^{i(1-\omega)x} + e^{-i(1+\omega)x} \right] dx$$

$$= \frac{1}{2\sqrt{2\pi}} \cdot \left\{ \left. \frac{e^{i(1-\omega)x}}{i(1-\omega)} \right|_{-\pi/2}^{\pi/2} + \left. \frac{e^{-i(1+\omega)x}}{-i(1+\omega)} \right|_{-\pi/2}^{\pi/2} \right\}$$

$$= \frac{1}{\cancel{2}\sqrt{2\pi}} \left\{ \frac{\sin((1-\omega)\pi/2)}{1-\omega} + \frac{\sin(1+\omega)\pi/2}{1+\omega} \right\}$$

Check the behaviour as $L \to \infty$

Q6. Now

$$\delta_L(x) = \frac{1}{2\pi} \int_{-L}^{L} e^{ikx} \, dx$$

$$= \frac{1}{2\pi} \left. \frac{e^{ikx}}{ik} \right|_{-L}^{L} = \left(\frac{L}{\cancel{2}\pi} \right) \frac{\sin(kL)}{kL}$$

(i) $$\int_{-\infty}^{\infty} \delta_L(k) \, dk = \frac{L}{\pi} \cdot \int_{-\infty}^{\infty} \frac{\sin(kL)}{(kL)} \, dk = \frac{1}{\pi} \int \frac{\sin t}{t} \, dt$$

contour integrals.

$$\lim_{L \to \infty} \int_{-\infty}^{\infty} \delta_L(k) \, dk = 1$$

$$\Rightarrow \int_{-\infty}^{\infty} \lim \delta_L(k) \, dk = \int_{-\infty}^{\infty} \delta(k) \, dk = 1$$

$$\frac{1}{\pi} \int_{aL}^{bL} \frac{\sin[t]}{t} \, dt \qquad t = xL$$

Q7. $$g(x) = A \exp\left[-\frac{(x-a)^2}{2\sigma^2}\right]$$

$$\int_{-\infty}^{\infty} g(x)\,dx = A \int_{-\infty}^{\infty} e^{-(x-a)^2/2\sigma^2}\,dx$$

put $t^2 = \frac{(x-a)^2}{2\sigma^2}$

$dt = \cancel{\frac{1}{2\sigma^2}} \frac{1}{\sqrt{2}\sigma} \cdot dx$.

$$= A \cdot \sqrt{2}\sigma \int_{-\infty}^{\infty} e^{-t^2} dt = A\sqrt{2\pi}\,\sigma$$

$$\Rightarrow \quad A = \frac{1}{\sqrt{2\pi}\,\sigma}$$

$$\langle x \rangle = \frac{1}{\sqrt{2\pi}\,\sigma} \cdot \int_{-\infty}^{\infty} (x)\, e^{-(x-a)^2/2\sigma^2} dx$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \underbrace{\int_{-\infty}^{\infty} (x-a)\, e^{-(x-a)^2/2\sigma^2} dx}_{0} + \frac{a}{\sqrt{2\pi\sigma}} \cdot \int_{-\infty}^{\infty} e^{-(x-a)^2/2\sigma^2} dx$$

$$= a.$$

$$\langle x^2 \rangle = \frac{1}{\sqrt{2\pi}\,\sigma} \int x^2 e^{-(x-a)^2/2\sigma^2} dx$$

put $t = \frac{(x-a)}{\sqrt{2}\,\sigma}$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} dt(\sqrt{2}\sigma) \cdot e^{-t^2} \cdot (\sqrt{2}\sigma t + a)^2$$

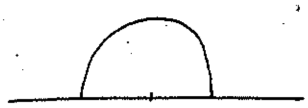
$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dt\, e^{-t^2} \left[2\sigma^2 t^2 + 2\sqrt{2}\sigma t a + a^2\right]$$

$$= \frac{1}{\sqrt{\pi}} \cdot \left[2\sigma^2 \cdot \underbrace{\Gamma\left(\tfrac{3}{2}\right)}_{\sqrt{\pi}/2} \quad 0 + a^2\sqrt{\pi}\right]$$

$$= \cancel{\frac{1}{\sqrt{\pi}}} \left[\sigma^2 + a^2\right]$$

8. $\psi(x,0) = A\left(a^2 - x^2\right)$, $\quad |x| \leq a$

$= 0 \quad$ otherwise



(a) $A^2 \int_{-a}^{a} \left(a^2 - x^2\right)^2 dx = A^2 \frac{16a^5}{15} = 1$

$\Rightarrow A = \sqrt{\frac{15}{16}a^5}$.

(b) $\langle x \rangle = 0 \qquad \langle x^2 \rangle = \frac{16a^7}{105} \cdot \frac{15}{16a^5} = \frac{a^2}{9}$

$\Rightarrow \sigma_x = \sqrt{\langle x^2 \rangle} = \cancel{\frac{4}{3\sqrt{5}}} \; \frac{a}{3}$.

(c) $\phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-a}^{a} A\left(a^2 - x^2\right) e^{-ipx/\hbar} \, dx$

$\cancel{= \frac{1}{\sqrt{2\pi\hbar}} A \cdot \left\{ a^2 \cdot \delta(p) - \int_{-\infty}^{\infty} x^2 e^{-ipx/\hbar} \, dp \right.}$

$= \frac{1}{\sqrt{2\pi\hbar}} A \cdot \left\{ a^2 \int_{-a}^{a} e^{-ipx/\hbar} \, dx \right.$

$\left. - \int_{-a}^{a} x^2 e^{-ipx/\hbar} \, dx \right.$

$= \frac{1}{\sqrt{2\pi\hbar}} A \left\{ 2a^2 \cdot \frac{\sin(pa/\hbar)}{p/\hbar} \right.$

$\left. - \int_{-a}^{a} x^2 \cos\left(\frac{px}{\hbar}\right) dx \right.$

$= \frac{A}{\sqrt{2\pi\hbar}} \left[2a^2 \frac{\sin\left(\frac{pa}{\hbar}\right)}{p/\hbar} + \frac{4\hbar^2 a}{p^2} \cos\left[\frac{pa}{\hbar}\right] \right.$

$\left. - \frac{2\hbar\left(p^2 a^2 - 2\hbar^2\right)}{p^3} \sin\left[\frac{pa}{\hbar}\right] \right\}$

$\frac{16}{3} a^3 \hbar^3 \pi$

$= \frac{A}{\sqrt{2\pi\hbar}} \left[\frac{4\hbar^2 a}{p^2} \cos\left[\frac{pa}{\hbar}\right] + \frac{4\hbar^3}{p^3} \sin\left(\frac{pa}{\hbar}\right) \right]$

$\langle p \rangle = 0$

$$\langle p^2 \rangle = \frac{\cancel{\hbar^2}}{2\cancel{\hbar\pi}} \, a \frac{\cancel{16}}{\cancel{3}} \cancel{\alpha^3} \cancel{\hbar^3} \cancel{\pi} \cdot \frac{\cancel{15}}{\cancel{16}\,\cancel{\alpha^8}\,a^2}$$

$$\frac{\cancel{32}}{\cancel{15}} \cancel{\alpha^8} \cancel{\hbar} \cancel{\pi} \cdot \frac{\cancel{15}}{16\cancel{\alpha^6}} \cdot \frac{1}{\cancel{2\hbar}}$$

$$= \frac{5\hbar^2 \cancel{\alpha}}{2a^2}$$

$$\sigma_p = \langle p^2 \rangle = \sqrt{\frac{5}{2}} \frac{\hbar}{a}$$

$$\sigma_x \sigma_p = \frac{\cancel{a}}{3} \frac{\hbar}{\cancel{a}} \cdot \sqrt{\frac{5}{2}} = \frac{\hbar\sqrt{5}}{3\sqrt{2}} = \frac{\sqrt{2 \cdot 5}}{3} \hbar$$

$$= 0.53\,\hbar > \frac{\hbar}{2}$$

9. $\psi(x,t) = B e^{iP_0 x/\hbar} e^{-|x|/2\Delta_x}$

(a) $B^2 \int_{-\infty}^{\infty} e^{-|x|/\cancel{2}\Delta_x} dx = B^2 2 \int_0^{\infty} e^{-x/\Delta_x} dx = B^2 2\Delta_x \Rightarrow B = \frac{1}{\sqrt{2\Delta_x}}$

(b) $A(P) = \frac{B}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \exp\left[-\frac{iPx}{\hbar} + \frac{iP_0 x}{\hbar} - \frac{|x|}{2\Delta_x}\right] dx$

$$= \frac{B}{\sqrt{2\pi\hbar}} \left[\int_0^{\infty} \exp\left[\left(\frac{i(P_0-P)}{\hbar} - \frac{1}{2\Delta_x}\right) x\right] dx + \int_{-\infty}^{0} \exp\left[\left(\frac{i(P_0-P)}{\hbar} + \frac{1}{2\Delta_x}\right) x\right] dx \right]$$

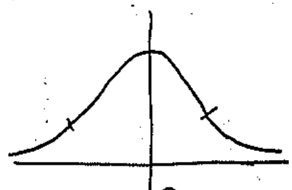
$y = -x$
$dy = -dx$

$$= \frac{B}{\sqrt{2\pi\hbar}} \left[\frac{-1}{\frac{i(P_0-P)}{\hbar} - \frac{1}{2\Delta_x}} + \int_0^{\infty} \exp\left[-\left(\frac{i(P_0-P)}{\hbar} + \frac{1}{2\Delta_x}\right) y\right] dy \right]$$

$$= \frac{B}{\sqrt{2\pi\hbar}} \left[\frac{\hbar}{\frac{\hbar}{2\Delta_x} - i(P_0-P)} + \frac{+1\hbar}{+\left[\frac{\hbar}{2\Delta_x} + i(P_0-P)\right]} \right]$$

$$= \frac{B\hbar}{\sqrt{2\pi\hbar}} \left[\frac{\hbar/\cancel{2}\Delta_x}{\frac{\hbar^2}{4\Delta_x^2} + (P_0-P)^2} \right]$$

$$= \frac{\hbar^2}{2\sqrt{\pi\hbar}\,\Delta_x^{3/2}} \cdot \frac{1}{\frac{\hbar^2}{4\Delta_x^2} + (P_0-P)^2}$$



P_0

$(P - P_0)^2 = \frac{\hbar^2}{4\Delta_x^2}$

$\sigma_p = \frac{\hbar}{\cancel{2}\Delta_x}$

$\sigma_x = \cancel{2}\Delta_x$

(P0) $A(p) = \left(\frac{1}{\sqrt{\pi}\,\Delta_p}\right)^{1/2} \exp\left[-\frac{(p-p_0)^2}{2\Delta_p^2}\right]$

Thus. $\psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \cdot \left(\frac{1}{\sqrt{\pi}\,\Delta_p}\right)^{1/2} \int_{-\infty}^{\infty} \exp\left[-\frac{(p-p_0)^2}{2\Delta_p^2} + i\frac{px}{\hbar} - \frac{iEt}{\hbar}\right] dp$

Write: $E(p) = \frac{p^2}{2m} = \frac{p_0^2}{2m} + \frac{p_0}{m}(p-p_0) + \frac{1}{2m}(p-p_0)^2$

Argument of Exponent.

$$= -\underbrace{\left[+\frac{1}{2\Delta_p^2} + \frac{it}{2m\hbar}\right]}_{\alpha}(p-p_0)^2 + \underbrace{\left[\frac{ix}{\hbar} - \frac{ip_0t}{m\hbar}\right]}_{\beta}(p-p_0) + \frac{ip_0x}{\hbar} - \frac{iE_0t}{\hbar}$$

$p-p_0=u$

$$\psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}}\left(\frac{1}{\sqrt{\pi}\,\Delta_p}\right)^{1/2} e^{i(p_0x-E_0t)/\hbar} \cdot \int_{-\infty}^{\infty} e^{-\alpha u^2 - \beta u}\, du$$

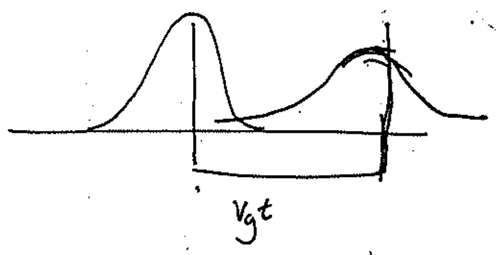
$$= \frac{1}{\sqrt{2\pi\hbar}}\left(\frac{1}{\sqrt{\pi}\,\Delta_p}\right)^{1/2} e^{i(p_0x-E_0t)/\hbar} \cdot \sqrt{\frac{\pi}{\alpha}} \cdot e^{+\beta^2/4\alpha}$$

$Re(\alpha) > 0$

$$= \left(\frac{\cancel{\pi}}{2\cancel{\pi}\hbar\,\alpha\sqrt{\pi}\,\Delta_p}\right)^{1/2} e^{i(p_0x-E_0t)/\hbar}\, e^{-(x-v_gt)^2/4\hbar^2\alpha}$$

$$(\psi^*\psi) = \frac{1}{2\sqrt{\pi}\hbar\Delta_p} \cdot \frac{1}{|\alpha|}\exp\left[\frac{(x-v_gt)^2}{4\hbar^2} \cdot \left[\frac{1}{\alpha} + \frac{1}{\bar{\alpha}}\right]\right]$$

$$\approx \frac{1}{|\alpha|}\exp\left[\frac{(x-v_gt)^2}{4\hbar^2\Delta_p^2}\frac{1}{|\alpha|^2}\right]$$



Compare with normalized gaussian fn then

$$\frac{1}{\sigma}e^{-\frac{x^2}{2\sigma^2}} \rightarrow \text{width} - \sigma$$

$|\alpha| = \left(\frac{1}{4\Delta_p^4} + \frac{t^2}{4m^2\hbar^2}\right)^{1/2}$ width grows with time