1. Let $\hat{\mathbf{S}}$ be the spin angular momentum operator with $s = \frac{1}{2}$.

   (a) Anticommutator of two operators $\hat{A}$ and $\hat{B}$, denoted by $\left[\hat{A}, \hat{B}\right]_+$ is defined as $\hat{A}\hat{B}+\hat{B}\hat{A}$. Prove that $\left[\hat{S}_x, \hat{S}_y\right]_+ = \left[\hat{S}_y, \hat{S}_z\right]_+ = \left[\hat{S}_z, \hat{S}_x\right]_+ = 0$.

   (b) Prove that $\hat{S}_i\hat{S}_j = \frac{i\hbar}{2}\hat{S}_k$, where $i, j, k = x$ or y or z, but only in cyclic order.

   (c) Prove $\hat{S}_i^2 = I$.

2. Let $\hat{\mathbf{S}}$ be the spin angular momentum operator of a particle with $s = \frac{1}{2}$. Find the eigenfunctions and eigenvalues of operators $\hat{S}_x$ and $\hat{S}_y$. If the state of the particle is

$$\Psi = \begin{bmatrix} \cos a \\ \sin a \, e^{ib} \end{bmatrix}$$

   where a, b are real constants, what is the probability that a measurement of $\hat{S}_y$ yields $-\hbar/2$.

3. Obtain the eigenvalues and corresponding normalized eigenvectors of $\hat{S}_n = \hat{\mathbf{n}} \cdot \hat{\mathbf{S}}$ for a particle of spin 1, where $\hat{\mathbf{n}}$ is a unit vector defined by polar coordinates (θ_0, ϕ_0).

4. Consider a spin-orbit interaction of a particle with spin 1. Write down the Hamiltonian in matrix form. Find all eigenvalues.

5. Two particles of the spin 1 have spin operators $\hat{\mathbf{S}}_1$ and $\hat{\mathbf{S}}_2$. Let $\hat{\mathbf{S}} = \hat{\mathbf{S}}_1 + \hat{\mathbf{S}}_2$. Find simultaneous eigenfunctions of $\hat{\mathbf{S}}^2$ and $\hat{S}_z$.

6. An electron is at rest in an oscillating magnetic field

$$\mathbf{B} = B_0 \cos(\omega t)\, \hat{k}$$

   where B_0 and ω are constants.

   (a) Construct the Hamiltonian of the system.

   (b) The state of the electron at $t = 0$ is given by 'up' state with respect to $x$axis, that is,

$$\Psi(0) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

   Find $\Psi(t)$ for $t > 0$. [Since the Hamiltonian is dependent on time, you must solve time-dependent Schrodinger equation directly.]

   (c) If you measure S_x, what is the probability that the result would be $-\hbar/2$?

   (d) What is the minimum value of B_0 that is required surely yield $-\hbar/2$ as a result of S_x measurement?

Tutorial 8

Q1. (a) $S_x = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $S_y = \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

$\therefore\ S_x S_y = \frac{\hbar^2}{4}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ and $S_y S_x = \frac{\hbar^2}{4}\begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$

$\Rightarrow\ [S_x, S_y]_+ = 0$. Prove other relations similarly.

(b) Now,

$$S_x S_y = \frac{\hbar^2}{4} i \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \frac{i\hbar}{2} S_z ,$$

(c) $S_x^2 = \frac{\hbar^2}{4} I$, $S_y^2 = \frac{\hbar^2}{4} I$. etc.

Q2. Eigenvalues of $\hat{S}_x$ and $\hat{S}_y$ are $-\frac{\hbar}{2}$ and $\frac{\hbar}{2}$

for $\hat{S}_x$: $\chi_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\chi_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\hat{S}_y$: $\xi_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}$ and $\xi_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} i \\ 1 \end{bmatrix}$

$$\text{Prob}(\hat{S}_y = -\tfrac{\hbar}{2}) = |\langle \xi_2 | \Psi \rangle|^2$$

$$= \frac{1}{2}\left| [-i, 1] \cdot \begin{bmatrix} \cos a \\ \sin a\ e^{ib} \end{bmatrix} \right|^2$$

$$= \frac{1}{2}\left| \left(-i\cos a + \sin a\, e^{ib} \right) \right|^2$$

$$= \frac{1}{2}\left(1 - \sin 2a \sin b \right)$$

Q4. Matrix of $L \cdot S$ $= \hbar \begin{bmatrix} L_z & L_-/\sqrt{2} & 0 \\ \frac{L_+}{\sqrt{2}} & 0 & \frac{L_-}{\sqrt{2}} \\ 0 & \frac{L_+}{\sqrt{2}} & -L_z \end{bmatrix}$

Let $\bar{J} = \bar{L} + \bar{S}$ then j quantum number can be $\ell+1, \ell, \ell-1$.

Since $\bar{L}\cdot\bar{S} = \frac{1}{2}(J^2 - L^2 - S^2)$

ev of $L \cdot S$ are given by $\frac{\hbar^2}{2}\left(j(j+1) - \ell(\ell+1) - 2 \right)$; $s = 1$.

Then evalus are

$j = \ell+1 \Rightarrow \lambda_1 = \ell\hbar^2$

$j = \ell \Rightarrow \lambda_2 = -\hbar^2$

$j = \ell-1 \Rightarrow \lambda_3 = -(\ell+1)\hbar^2$

Tutorial 8

Q3. Given

$$S_x = \frac{\hbar}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad S_y = \frac{\hbar}{\sqrt{2}}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \quad S_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Let $\hat{n} = \hat{i}\sin\theta\cos\phi + \hat{j}\sin\theta\sin\phi + \hat{k}\cos\theta$

Thus $\vec{S}\cdot\hat{n} = \sin\theta\cos\phi\,\hat{S}_x + \sin\theta\sin\phi\,\hat{S}_y + \cos\theta\,\hat{S}_z = \hat{S}_{\hat{n}}$

$$= \hbar\begin{bmatrix} \cos\theta & \frac{\sin\theta}{\sqrt{2}}e^{-i\phi} & 0 \\ \frac{\sin\theta}{\sqrt{2}}e^{i\phi} & 0 & \frac{\sin\theta}{\sqrt{2}}e^{-i\phi} \\ 0 & \frac{\sin\theta}{\sqrt{2}}e^{i\phi} & -\cos\theta \end{bmatrix}$$

Eigenvalues of $\hat{S}_n/\hbar$ are given by

$$\det\left(\frac{\hat{S}_n}{\hbar} - \lambda I\right) = 0$$

$$\Rightarrow (\cos\theta - \lambda)\left[(-\lambda)(-\cos\theta - \lambda) - \frac{\sin^2\theta}{2}\right] + - \frac{\sin^2\theta}{2}(-\cos\theta - \lambda) = 0$$

$\Rightarrow \lambda(1-\lambda^2) = 0 \quad \Rightarrow \quad \lambda = 0, 1$ or -1. Let $\lambda_1 = 1, \lambda_2 = 0, \lambda_3 = -1$.

For λ_1, let $\hat{S}_n\psi_1 = \lambda_1\psi_1$ with $\psi_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$

$$(\cos\theta - 1)x_1 + \frac{\sin\theta}{\sqrt{2}}e^{-i\phi}y_1 = 0$$

$$\frac{\sin\theta}{\sqrt{2}}e^{i\phi}x_1 + (-1)y_1 + \frac{\sin\theta}{\sqrt{2}}e^{-i\phi}z_1 = 0$$

$$\frac{\sin\theta}{\sqrt{2}}e^{i\phi}y_1 + (\cos\theta + 1)z_1 = 0$$

Let $y_1 = A \quad \Rightarrow \quad x_1 = A\frac{\sin\theta}{\sqrt{2}(1-\cos\theta)}e^{-i\phi} = \frac{A}{\sqrt{2}}\cot\frac{\theta}{2}e^{-i\phi}$

and $z_1 = A\frac{\sin\theta}{\sqrt{2}(1+\cos\theta)}e^{i\phi} = \frac{A}{\sqrt{2}}\tan\frac{\theta}{2}e^{i\phi}$

Now, normalize,

$$|x_1|^2 + |y_1|^2 + |z_1^2| = 1$$

$$\Rightarrow A^2\left[1 + \frac{\sin^2\theta}{2(1-\cos\theta)^2} + \frac{\sin^2\theta}{2(1+\cos\theta)^2}\right] = 1$$

$$\Rightarrow A^2\left[1 + \frac{\sin^2\theta}{2}\left(\frac{2 + 2\cos^2\theta}{(1-\cos^2\theta)^2}\right)\right] = 1$$

$$\Rightarrow \quad A^2 \left[1 + \frac{(1+\cos^2\theta)}{\sin^2\theta} \right] = 1$$

$$\Rightarrow \quad A^2 \frac{2}{\sin^2\theta} = 1$$

$$\Rightarrow \quad A = \frac{\sin\theta}{\sqrt{2}}$$

Then $y_1 = \dfrac{\sin\theta}{\sqrt{2}}$, $x_1 = \dfrac{1}{2} \cdot 2\sin\theta/2 \cdot \cos\theta/2 \cdot \dfrac{\cos\theta/2}{\sin\theta/2}\, e^{-i\phi} = \cos^2\dfrac{\theta}{2} \cdot e^{-i\phi}$

$$z_1 = \sin^2\frac{\theta}{2}\, e^{+i\phi}$$

Thus ~~Ev~~

$$\begin{array}{ccc}
\text{ev} & \text{estate} & \theta = 0 \\
\hbar & \begin{bmatrix} \cos^2\theta/2\, e^{-i\phi} \\ \sin\theta/\sqrt{2} \\ \sin^2\frac{\theta}{2}\, e^{i\phi} \end{bmatrix} & \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\
0 & \begin{bmatrix} -\frac{\sin\theta}{\sqrt{2}}\, e^{-i\phi} \\ \cos\theta \\ \frac{\sin\theta}{\sqrt{2}}\, e^{i\phi} \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\
-\hbar & \begin{bmatrix} \sin^2\frac{\theta}{2}\, e^{-i\phi} \\ -\sin\theta/\sqrt{2} \\ \cos^2\frac{\theta}{2}\, e^{i\phi} \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\end{array}$$

Q5. Let $S = S_1 + S_2$. given $s_1 = s_2 = 1$

Let $S_1^2 \chi_{s_1,m_1} = \hbar^2 s_1(s_1+1)\chi_{s_1,m_1}$

and $S_{1z}\chi_{s_1,m_1} = \hbar m_1 \chi_{s_1,m_1}$

Similarly ~~let~~ ξ_{s_2,m_2} be eigenstates of S_2^2 and S_{2z}

The product states will be denoted by

$$|m_1 m_2\rangle = \chi_{s_1,m_1}\cdot\xi_{s_2,m_2}$$ since S_1 and S_2 are fixed.

Let $\psi_{s,M}$ be eigenstates of S^2, S_z, S_1^2 and S_2^2.

possible values for s are $2, 1, 0$.

Then,

$$\underline{\psi_{2,0} = |1,1\rangle}$$

Now.

$$S_-\psi_{2,0} = (S_{1-} + S_{2-})|1,1\rangle$$

$$\Rightarrow \quad 2\psi_{2,1} = \sqrt{2}\,|1,0\rangle + \sqrt{2}\,|0,1\rangle$$

$$\Rightarrow \quad \underline{\psi_{2,1} = \frac{1}{\sqrt{2}}\left(|1,0\rangle + |0,1\rangle\right)}$$

$$S_-\psi_{2,1} = \frac{1}{\sqrt{2}}(S_{1-} + S_{2-})\left(|1,0\rangle + |0,1\rangle\right)$$

$$\Rightarrow \quad \sqrt{6}\,\psi_{2,0} = \frac{1}{\sqrt{2}}\left(\sqrt{2}\,|0,0\rangle + \sqrt{2}\,|1,\bar{1}\rangle + \sqrt{2}\,|1,\bar{1}\rangle + \sqrt{2}\,|0,0\rangle\right)$$

$$\Rightarrow \quad \underline{\psi_{2,0} = \frac{1}{\sqrt{6}}\left(|1,\bar{1}\rangle + 2\,|0,0\rangle + |\bar{1},1\rangle\right)}$$

Now $\psi_{1,1}$ must be orthogonal to $\psi_{2,1}$

$$\Rightarrow \quad \underline{\psi_{1,1} = \frac{1}{\sqrt{2}}\left(|1,0\rangle - |0,1\rangle\right)}$$

and

$$S_-\psi_{1,1} = \frac{1}{\sqrt{2}}\left(\sqrt{2}\right)\left(|1,\bar{1}\rangle + |0,0\rangle - |0,0\rangle - |\bar{1},1\rangle\right)$$

$$\underline{\psi_{1,0} = \frac{1}{\sqrt{2}}\left(|1,\bar{1}\rangle - |\bar{1},1\rangle\right)}$$

Finally $\psi_{0,0}$ must be orthogonal to $\psi_{1,0}$ and $\psi_{2,0}$

$$\underline{\psi_{0,0} = \frac{1}{\sqrt{3}}\left(|1,\bar{1}\rangle - |0,0\rangle + |\bar{1},1\rangle\right)}$$

Tutorial 8

Q6. $\hat{H} = -\vec{\mu}\cdot\vec{B}$

where $\vec{\mu} = -g_e \mu_B \vec{S}$. since $\vec{B} = B_0 \cos(\omega t)\,\hat{k}$

(a) $\Rightarrow$ $\hat{H} = (g_e \mu_B B_0)\cos(\omega t)\, S_z$

$$= \alpha \cos(\omega t)\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \alpha = \frac{g_e \mu_B B_0 \hbar}{2}.$$

Let $\Psi(t) = \begin{bmatrix} f(t) \\ g(t) \end{bmatrix}$ with $\Psi(0) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Schrödinger Eq.

$$i\hbar \frac{d}{dt}\Psi(t) = \hat{H}\Psi(t)$$

$$\Rightarrow \begin{bmatrix} f'(t) \\ g'(t) \end{bmatrix} = \frac{\alpha\cos(\omega t)}{i\hbar}\begin{bmatrix} f(t) \\ -g(t) \end{bmatrix}$$

$$\Rightarrow f'(t) = \frac{\alpha\cos(\omega t)}{i\hbar} f(t) \Rightarrow f(t) = \frac{1}{\sqrt{2}} e^{-i\frac{\alpha}{\hbar\omega}\sin(\omega t)}$$

and $$g'(t) = -\frac{\alpha\cos\omega t}{i\hbar} g(t) \Rightarrow g(t) = \frac{1}{\sqrt{2}} e^{i\frac{\alpha}{\hbar\omega}\sin(\omega t)}$$

$\Rightarrow$

(c) $$P\left(\hat{S}_x = -\frac{\hbar}{2}\right) = \left|\left\langle \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \Psi(t)\right\rangle\right|^2$$

$$= \left|\frac{1}{\sqrt{2}}[1, -1]\cdot\frac{1}{\sqrt{2}}\begin{bmatrix} \exp[-i\frac{\alpha}{\hbar\omega}\sin(\omega t)] \\ \exp[i\frac{\alpha}{\hbar\omega}\sin(\omega t)] \end{bmatrix}\right|^2$$

$$= \left[\sin\left(\frac{\alpha}{\hbar\omega}\sin(\omega t)\right)\right]^2$$

(d) $P(\hat{S}_x = -\frac{\hbar}{2})$ will be equal to 1 if

$$\frac{\alpha}{\hbar\omega} \geq \frac{\pi}{2}$$

$$\Rightarrow \boxed{B_0 \geq \frac{\pi\omega}{g_e\mu_B}}$$