1. Let $l \geqslant m \geqslant 0$.
   (a) Show that $P_l^m(-x) = (-1)^{l-m} P_l^m(x)$.
   (b) If the spherical coordinates of a vector **r** are (r, θ, ϕ), what are the sperical coordinates of $-\mathbf{r}$?
   (c) Show that $Y_{l,m}$ has a parity $(-1)^l$ under $\mathbf{r} \to -\mathbf{r}$ transformation.

2. Show that $\mathbf{L}_- = \hbar e^{-i\phi}\left(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)$. Also show that $\mathbf{L}_- Y_{l,m} = \hbar\sqrt{l\,(l+1) - m\,(m-1)}\,Y_{l,m-1}$. Use recurrence relation 6.97d given on page 282 of Bransden.

3. Just as $Y_{l,m}$ is a an eigenfunctions of $\mathbf{L}^2$ and $\mathbf{L}_z$, let $Z_{l,n}$ be an eigenfunction of $\mathbf{L}^2$ and $\mathbf{L}_x$ that is

$$\begin{aligned}
\mathbf{L}^2 Z_{l,n} &= l(l+1)\hbar^2 Z_{l,n} \\
\mathbf{L}_x Z_{l,n} &= n\hbar Z_{l,n}.
\end{aligned}$$

   We already know the eigenvalues of $\mathbf{L}^2$, that is $l = 0, 1, 2, \ldots$.
   (a) What are allowed values for n for given l?
   (b) Show that $\langle Z_{l,n}, Y_{l',m}\rangle = 0$ if $l \neq l'$.
   (c) Show that $Z_{l,n}$ can be expressed in terms of $Y_{l,m}$, that is

$$Z_{l,n} = \sum_{m=-l}^{l} C_{n,m} Y_{l,m}.$$

   (d) Find $Z_{1,n}$ in terms of $Y_{1,m}$ explicitly.

4. Let the state of a particle constrained to move on a sphere, be $Y_{1,0}$. What are the possible results of measurement of L_x? What is the probability associated with each outcome?

5. The state of a particle constrained to move on a sphere is

$$\Psi(\theta, \phi, t = 0) = \frac{1}{\sqrt{4\pi}}\left(e^{i\phi}\sin\theta + \cos\theta\right)$$

   (a) What are the probabilities for the various results of the measurement of $\mathbf{L}_z$ at $t = 0$? What about at $t > 0$?
   (b) What is the expectation value of $\mathbf{L}_z$ at $t = 0$?

6. Let $|j, m\rangle$ be the eigenstate of the angular momentum operators $\mathbf{J}^2$ and $\mathbf{J}_z$ with eigenvalues $j(j+1)\hbar^2$ and $m\hbar$ respectively. Obtain a matrix representation of operators $\mathbf{J}_x$, $\mathbf{J}_y$, $\mathbf{J}_z$, $\mathbf{J}_+$, $\mathbf{J}_-$ and $\mathbf{J}^2$ using $|j, m\rangle$ as the basis for $j = 3/2$.

7. Show that

$$\mathbf{J}_\pm |j, m\rangle = \sqrt{j(j+1) - m(m \pm 1)}\,|j, m \pm 1\rangle$$

Tutorial 7

Q1. $l \geq m \geq 0$,

(a) By Rodrigue's Formula

$$P_l^m(x) = (1-x^2)^{m/2} \frac{d^m}{dx^m} P_l(x)$$

$(1-x^2)^{m/2}$ is even. $P_l(x)$ is polynomial with parity $(-1)^l$ with every derivative parity ~~ch~~ is multiplied by -1. Thus parity of $P_l^m(x)$ is $(-1)^{l-m}$.

(b) With $\vec{r} = (r,\theta,\phi)$, $-\vec{r} = (r, \pi-\theta, \pi+\phi)$

(c) Spherical Harmonics

$$Y_{lm} = (-1)^m N_n P_l^m(\cos\theta)\, e^{im\phi}$$

$$\begin{aligned}
\Rightarrow \quad Y_{lm}(-\hat{r}) &= Y_{lm}(\pi-\theta, \pi+\phi) \\
&= (-1)^m N_n P_l^m(+\cos(\pi-\theta))\, e^{im(\phi+\pi)} \\
&= (-1)^m N_n P_l^m(-x)\, e^{im\phi} (-1)^m \\
&= (-1)^m N_n (-1)^{l-m} P_l^m(\cos\theta)\, e^{im\phi} (-1)^m \\
&= (-1)^l Y_{lm}(\theta,\phi).
\end{aligned}$$

Q2. From $L_x = (-i\hbar)\left[-\sin\phi \frac{\partial}{\partial\theta} - \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right]$

and $L_y = (-i\hbar)\left[\cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right]$

It is easy to see that

$$L_- = L_x - iL_y = \hbar e^{-i\phi}\left[-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right]$$

Then

$$\begin{aligned}
L_- Y_{lm} &= N_{lm}\frac{\hbar e^{-i\phi}}{\sin\theta}\left[-\sin\theta\frac{\partial}{\partial\theta} + i\cos\theta\frac{\partial}{\partial\phi}\right] P_l^m(\cos\theta)\, e^{im\phi} \\
&= N_{lm}\frac{\hbar e^{-i\phi}}{\sin\theta}\left[-\sin\theta\frac{\partial}{\partial\theta} - m\cos\theta\right] P_l^m(\cos\theta)\, e^{im\phi}
\end{aligned}$$

Now: $$(1-x^2)\frac{d}{dx}P_\ell^m(x) = -(1-x^2)^{1/2}(\ell+m)\cdot(\ell-m+1)P_\ell^{m-1}(x) + mx P_\ell^m(x)$$

put $$x=\cos\theta \Rightarrow \frac{d}{d\theta} = \frac{dx}{d\theta}\cdot\frac{d}{dx} = -\sin\theta\frac{d}{dx}$$

$$\Rightarrow \quad \sin^2\theta\cdot\left(-\frac{1}{\sin\theta}\frac{d}{d\theta}\right)P_\ell^m(\cos\theta) = -\sin\theta\,(\ell+m)(\ell-m+1)P_\ell^{m-1} + mxP_\ell^m$$

Substituting back in $L_- Y_{\ell m}$,

$$L_- Y_{\ell m} = N_{\ell m}\frac{\hbar e^{-i\phi}}{\sin\theta}\left[-\sin\theta\,(\ell+m)(\ell-m+1)P_\ell^{m-1}\right]e^{im\phi}$$

$$= (-1)^m\left[\frac{(2\ell+1)}{4\pi}\cdot\frac{(\ell-m)!}{(\ell+m)!}\right]^{1/2}(-\hbar)(\ell+m)(\ell-m+1)P_\ell^{m-1}e^{i(m-1)\phi}$$

$$= (-1)^{m-1}\left[\frac{2\ell+1}{4\pi}\cdot\frac{(\ell-m+1)!}{(\ell+m-1)!}\right]^{1/2}P_\ell^{m-1}e^{i(m-1)\phi}\cdot\hbar\sqrt{(\ell+m)(\ell-m+1)}$$

$$= \hbar\sqrt{(\ell+m)(\ell-m+1)}\cdot Y_{\ell m}.$$

Q3. Given

$$\left.\begin{aligned} L^2 Z_{\ell,n} &= \ell(\ell+1)\hbar^2 Z_{\ell n} \\ L_x Z_{\ell,n} &= n\hbar Z_{\ell,n}\end{aligned}\right\} \quad \ell=0,1,\cdots \text{ but we don't know } n \text{ yet.}$$

(a) It is easy to see that eigenvalues of L_x must be $n\hbar$ s.t.

$$-\ell\le n\le\ell \quad \text{and } n \text{ integral.}$$

(b) $Z_{\ell,n}$ and $Y_{\ell',m}$ are eigenfunctions of L^2 (Hermitian) operator then

$$\langle Z_{\ell,n}, Y_{\ell',m}\rangle = 0 \quad \text{if } \ell\ne\ell'$$

(c) Now every function, including $Z_{\ell,n}$ can be written in terms of harmonics.

$$Z_{\ell,n} = \sum_{\ell'}\sum_{m'} a^{\ell n}_{\ell' m'} Y_{\ell' m'}$$

where

$$a^{ln}_{l'm'} = \langle z_{l,n} , Y_{l'm'} \rangle^*$$

but $\quad a^{ln}_{l'm'} = 0 \quad \text{if} \quad l \neq l'$

$$\Rightarrow \quad z_{ln} = \sum_{m'} a^{ln}_{lm'} Y_{lm'}$$

$$= \sum_m C_{n,m} Y_{lm} \qquad \text{let } a^{ln}_{lm} = C_{n,m}.$$

(d) Now,

$$z_{1,1} = \sum_m C_{1,m} Y_{1m}$$

but $\quad L_x z_{1,1} = \hbar z_{11} = \cancel{L_x} \left(C_{1,1} Y_{1,1} + C_{1,0} Y_{1,0} + C_{1,\bar{1}} Y_{1,\bar{1}} \right)$

$$\hbar \left(C_{11} Y_{11} + C_{10} Y_{10} + C_{1\bar{1}} Y_{1\bar{1}} \right) = \frac{1}{2} (L_+ + L_-) \left[C_{1,1} Y_{1,1} + C_{1,0} Y_{1,0} + C_{1\bar{1}} Y_{1\bar{1}} \right]$$

$$= \frac{\hbar}{2} \left[C_{11} \left(\sqrt{2}\, Y_{10} \right) + C_{10} \sqrt{2} \left(Y_{11} + Y_{1\bar{1}} \right) + C_{1\bar{1}} Y_{10} \sqrt{2} \right]$$

$\Rightarrow$

$$\hbar C_{11} = \hbar C_{10} \sqrt{2}/2$$

$$\hbar C_{10} = \frac{\hbar}{\sqrt{2}} \left(C_{11} + C_{1\bar{1}} \right)$$

$$\hbar C_{1\bar{1}} = \hbar C_{10}/\sqrt{2}$$

$$\Rightarrow C_{10} = \frac{1}{\sqrt{2}} \quad \text{and} \quad C_{11} = C_{1\bar{1}} = \frac{1}{2}.$$

$$\Rightarrow \quad z_{11} = \frac{1}{2} \left[Y_{11} + \sqrt{2}\, Y_{1,0} + Y_{1\bar{1}} \right]$$

$$z_{10} = \frac{1}{\sqrt{2}} \left[Y_{11} - Y_{1\bar{1}} \right]$$

$$z_{1\bar{1}} = \frac{1}{2} \left[Y_{11} - \sqrt{2}\, Y_{10} + Y_{1\bar{1}} \right]$$

Q4. Continuing from Q3:

possible results for measurement of L_x are $\hbar, 0, -\hbar$.

$$P(L_x = \hbar) = |\langle Z_{11}, Y_{1,0}\rangle|^2 = \frac{1}{2}$$

$$P(L_x = 0) = |\langle Z_{1,0}, Y_{1,0}\rangle|^2 = 0$$

$$P(L_x = -\hbar) = |\langle Z_{1,-1}, Y_{10}\rangle|^2 = \frac{1}{2}.$$

Q5. Given: $$\psi(\theta,\phi,t=0) = \frac{1}{\sqrt{4\pi}}\left(e^{i\phi}\sin\theta + \cos\theta\right)$$

$$= \frac{1}{\sqrt{4\pi}}\left(-\sqrt{\frac{8\pi}{3}}\,Y_{11} + \sqrt{\frac{4\pi}{3}}\,Y_{10}\right)$$

$$= \frac{1}{\sqrt{3}}\left(Y_{10} - \sqrt{2}\,Y_{11}\right)$$

(a) $$\left.\begin{aligned} P(L_z = \hbar) &= \frac{2}{3} \\ P(L_z = 0) &= \frac{1}{3} \\ P(L_z = -\hbar) &= 0 \end{aligned}\right\} \text{ at all times.}$$

(b) $$\langle L_z\rangle = \frac{2\hbar}{3}.$$

Q6. Clearly $J^2 = \frac{15}{4}\hbar^2 I_{4\times4}$ and

$$J_z = \hbar\begin{bmatrix} 3/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & -\frac{3}{2} \end{bmatrix}$$

Now $$J_+ = \begin{bmatrix} 0 & \sqrt{3} & & \\ 0 & 0 & 2 & \\ 0 & 0 & 0 & 2\sqrt{3} \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } J_- = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix}$$

Q7. Since $J_-J_+ = J^2 - J_z(J_z+1)$

and $J_+|jm\rangle = c_m|j, m+1\rangle$

Then

$$\langle jm|J_-J_+|jm\rangle = c_m^2 \langle j, m+1|j, m+1\rangle$$

$$\Rightarrow \quad \big(j(j+1) - m(m+1)\big) = c_m^2$$

$$\Rightarrow \quad c_m = \sqrt{j(j+1) - m(m+1)}$$

Similarly $J_+J_- = J^2 - J_z(J_z-1)$, $J_-|jm\rangle = d_m|j, m-1\rangle$

gives

$$d_m = \sqrt{j(j+1) - m(m-1)}.$$