1. *[G 5.2] Find and sketch the trajectory of the particle shown in Fig., if it starts at the origin with velocity*

   (a) $\mathbf{v}(0) = (E/B)\,\hat{\mathbf{y}}$,
   (b) $\mathbf{v}(0) = (E/2B)\,\hat{\mathbf{y}}$,
   (c) $\mathbf{v}(0) = (E/B)\,(\hat{\mathbf{y}} + \hat{\mathbf{z}})$.

The general solution is

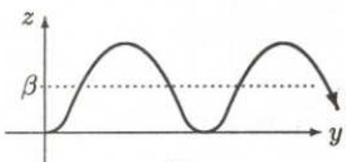
$$y(t) = C_1\cos(\omega t) + C_2\sin(\omega t) + \frac{E}{B}t + C_3; \quad z(t) = C_2\cos(\omega t) - C_1\sin(\omega t) + C_4.$$

(a) $y(0) = z(0) = 0$; $\dot{y}(0) = E/B$; $\dot{z}(0) = 0$. Use these to determine C_1, C_2, C_3, and C_4.
$y(0) = 0 \Rightarrow C_1 + C_3 = 0$; $\dot{y}(0) = \omega C_2 + E/B = E/B \Rightarrow C_2 = 0$; $z(0) = 0 \Rightarrow C_2 + C_4 = 0 \Rightarrow C_4 = 0$;
$\dot{z}(0) = 0 \Rightarrow C_1 = 0$, and hence also $C_3 = 0$. So $\boxed{y(t) = Et/B;\ z(t) = 0.}$ Does this make sense? The magnetic force is $q(\mathbf{v}\times\mathbf{B}) = -q(E/B)B\,\hat{\mathbf{z}} = -q\mathbf{E}$, which exactly cancels the electric force; since there is no net force, the particle moves in a straight line at constant speed.

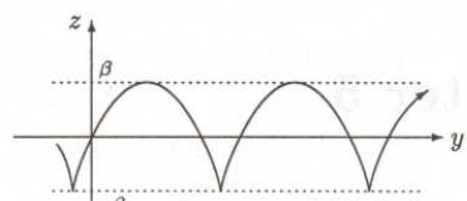
(b) Assuming it starts from the origin, so $C_3 = -C_1$, $C_4 = -C_2$, we have $\dot{z}(0) = 0 \Rightarrow C_1 = 0 \Rightarrow C_3 = 0$;
$\dot{y}(0) = \dfrac{E}{2B} \Rightarrow C_2\omega + \dfrac{E}{B} = \dfrac{E}{2B} \Rightarrow C_2 = -\dfrac{E}{2\omega B} = -C_4$; $y(t) = -\dfrac{E}{2\omega B}\sin(\omega t) + \dfrac{E}{B}t$;
$z(t) = -\dfrac{E}{2\omega B}\cos(\omega t) + \dfrac{E}{2\omega B}$, or $\boxed{y(t) = \dfrac{E}{2\omega B}\left[2\omega t - \sin(\omega t)\right];\ z(t) = \dfrac{E}{2\omega B}\left[1 - \cos(\omega t)\right].}$ Let $\beta \equiv E/2\omega B$.
Then $y(t) = \beta\left[2\omega t - \sin(\omega t)\right]$; $z(t) = \beta\left[1 - \cos(\omega t)\right]$; $(y - 2\beta\omega t) = -\beta\sin(\omega t)$, $(z - \beta) = -\beta\cos(\omega t) \Rightarrow$
$(y - 2\beta\omega t)^2 + (z - \beta)^2 = \beta^2$. This is a circle of radius β whose center moves to the right at constant speed: $y_0 = 2\beta\omega t$; $z_0 = \beta$.

(c) $\dot{z}(0) = \dot{y}(0) = \dfrac{E}{B} \Rightarrow -C_1\omega = \dfrac{E}{B} \Rightarrow C_1 = -C_3 = -\dfrac{E}{\omega B}$; $C_2\omega + \dfrac{E}{B} = \dfrac{E}{B} \Rightarrow C_2 = C_4 = 0$.

$y(t) = -\dfrac{E}{\omega B}\cos(\omega t) + \dfrac{E}{B}t + \dfrac{E}{\omega B}$; $z(t) = \dfrac{E}{\omega B}\sin(\omega t)$. $\boxed{y(t) = \dfrac{E}{\omega B}\left[1 + \omega t - \cos(\omega t)\right];\ z(t) = \dfrac{E}{\omega B}\sin(\omega t).}$
Let $\beta \equiv E/\omega B$; then $[y - \beta(1+\omega t)] = -\beta\cos(\omega t)$, $z = \beta\sin(\omega t)$; $[y - \beta(1+\omega t)]^2 + z^2 = \beta^2$. This is a circle of radius β whose center is at $y_0 = \beta(1+\omega t)$, $z_0 = 0$.



(b)



(c)

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2. *[G 5.4] Suppose that the magnetic field in some region has the form*

$$\mathbf{B} = kz\hat{\mathbf{x}}$$

(where k is a constant). Find the force on a square loop (side a), lying in the yz plane and centered at the origin, if it carries a current I, flowing counterclockwise, when you look down the x axis.

Suppose I flows counterclockwise (if not, change the sign of the answer). The force on the left side (toward the left) cancels the force on the right side (toward the right); the force on the top is $IaB = Iak(a/2) = Ika^2/2$, (pointing upward), and the force on the bottom is $IaB = -Ika^2/2$ (also upward). So the net force is $\mathbf{F} = \boxed{Ika^2\,\hat{\mathbf{z}}.}$

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3. *[G 5.13] A steady current I flows down a long cylindrical wire of radius a (Fig.). Find the magnetic field, both inside and outside the wire, if*

   (a) *The current is uniformly distributed over the outside surface of the wire.*

   (b) *The current is distributed in such a way that J is proportional to s, the distance from the axis.*

(a) $\oint \mathbf{B}\cdot d\mathbf{l} = B\,2\pi s = \mu_0 I_{\text{enc}} \Rightarrow \boxed{\mathbf{B} = \left\{\begin{array}{ll} 0, & \text{for } s < a; \\ \dfrac{\mu_0 I}{2\pi s}\,\hat{\boldsymbol{\phi}}, & \text{for } s > a. \end{array}\right\}}$

(b) $J = ks;\ I = \int_0^a J\,da = \int_0^a ks(2\pi s)\,ds = \dfrac{2\pi k a^3}{3} \Rightarrow k = \dfrac{3I}{2\pi a^3}.\ \ I_{\text{enc}} = \int_0^s J\,da = \int_0^s k\bar{s}(2\pi\bar{s})\,d\bar{s} = \dfrac{2\pi k s^3}{3} = I\dfrac{s^3}{a^3}$, for $s < a$; $I_{\text{enc}} = I$, for $s > a$. So $\boxed{\mathbf{B} = \left\{\begin{array}{ll} \dfrac{\mu_0 I s^2}{2\pi a^3}\,\hat{\boldsymbol{\phi}}, & \text{for } s < a; \\ \dfrac{\mu_0 I}{2\pi s}\,\hat{\boldsymbol{\phi}}, & \text{for } s > a. \end{array}\right\}}$

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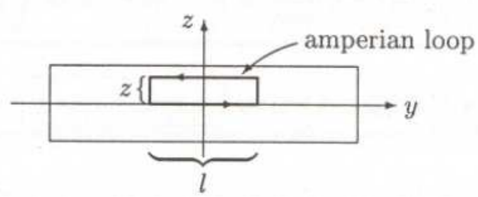
4. *[G 5.14] A thick slab extending from $z = -a$ to $z = +a$ carries a uniform volume current $\mathbf{J} = J\hat{\mathbf{x}}$ (Fig.). Find the magnetic field, as a function of z, both inside and outside the slab.*

By the right-hand-rule, the field points in the $-\hat{\mathbf{y}}$ direction for $z > 0$, and in the $+\hat{\mathbf{y}}$ direction for $z < 0$. At $z = 0, B = 0$. Use the amperian loop shown:
$\oint \mathbf{B}\cdot d\mathbf{l} = Bl = \mu_0 I_{\text{enc}} = \mu_0 lzJ \Rightarrow \boxed{\mathbf{B} = -\mu_0 Jz\,\hat{\mathbf{y}}}$ $(-a < z < a)$. If $z > a, I_{\text{enc}} = \mu_0 laJ$,

so $\boxed{\mathbf{B} = \left\{\begin{array}{ll} -\mu_0 Ja\,\hat{\mathbf{y}}, & \text{for } z > +a; \\ +\mu_0 Ja\,\hat{\mathbf{y}}, & \text{for } z > -a. \end{array}\right\}}$



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5. *[G 5.23] What current density would produce the vector potential, $\mathbf{A} = k\hat{\boldsymbol{\phi}}$ (where k is a constant), in cylindrical coordinates?*

$$A_\phi = k \Rightarrow \mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A} = \frac{1}{s}\frac{\partial}{\partial s}(sk)\,\hat{\mathbf{z}} = \frac{k}{s}\,\hat{\mathbf{z}};\ \mathbf{J} = \frac{1}{\mu_0}(\boldsymbol{\nabla}\times\mathbf{B}) = \frac{1}{\mu_0}\left[-\frac{\partial}{\partial s}\left(\frac{k}{s}\right)\right]\hat{\boldsymbol{\phi}} = \boxed{\frac{k}{\mu_0 s^2}\,\hat{\boldsymbol{\phi}}.}$$

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6. *[G 5.24] If $\mathbf{B}$ is* uniform, *show that $\mathbf{A}(\mathbf{r}) = -\frac{1}{2}(\mathbf{r}\times\mathbf{B})$. That is, check that $\boldsymbol{\nabla}\cdot\mathbf{A} = 0$ and $\boldsymbol{\nabla}\times\mathbf{A} = \mathbf{B}$. Is this result unique, or are there other functions with the same divergence and curl?*

$\nabla \cdot \mathbf{A} = -\frac{1}{2}\nabla \cdot (\mathbf{r} \times \mathbf{B}) = -\frac{1}{2}[\mathbf{B} \cdot (\nabla \times \mathbf{r}) - \mathbf{r} \cdot (\nabla \times \mathbf{B})] = 0$, since $\nabla \times \mathbf{B} = 0$ (**B** is uniform) and $\nabla \times \mathbf{r} = 0$ (Prob. 1.62). $\nabla \times \mathbf{A} = -\frac{1}{2}\nabla \times (\mathbf{r} \times \mathbf{B}) = -\frac{1}{2}[(\mathbf{B} \cdot \nabla)\mathbf{r} - (\mathbf{r} \cdot \nabla)\mathbf{B} + \mathbf{r}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{r})]$. But $(\mathbf{r} \cdot \nabla)\mathbf{B} = 0$ and $\nabla \cdot \mathbf{B} = 0$ (since **B** is uniform), and $\nabla \cdot \mathbf{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1 + 1 + 1 = 3$. Finally, $(\mathbf{B} \cdot \nabla)\mathbf{r} = \left(B_x\frac{\partial}{\partial x} + B_y\frac{\partial}{\partial y} + B_z\frac{\partial}{\partial z}\right)(x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}) = B_x\,\hat{\mathbf{x}} + B_y\,\hat{\mathbf{y}} + B_z\,\hat{\mathbf{z}} = \mathbf{B}$. So $\nabla \times \mathbf{A} = -\frac{1}{2}(\mathbf{B} - 3\mathbf{B}) = \mathbf{B}$. qed

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7. *[G 5.35] A phonograph record of radius R, carrying a uniform surface charge σ, is rotating at constant angular velocity ω. Find its magnetic dipole moment.*

For a ring, $m = I\pi r^2$. Here $I \to \sigma v\, dr = \sigma\omega r\, dr$, so $m = \int_0^R \pi r^2 \sigma\omega r\, dr = \boxed{\pi\sigma\omega R^4/4}$.

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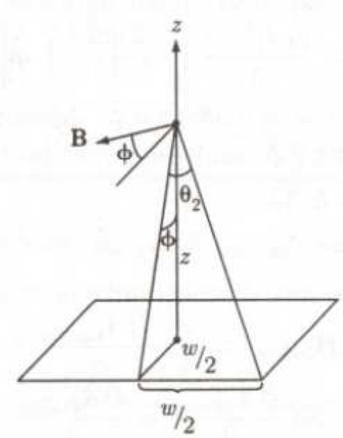
8. *[G 5.37] Find the exact magnetic field a distance z above the center of a square loop of side w, carrying a curent I. Verify that it reduces to the field of a dipole, with the appropriate dipole moment, when $z \gg w$.*

The field of one side is given by Eq. 5.35, with $s \to \sqrt{z^2 + (w/2)^2}$ and $\sin\theta_2 = -\sin\theta_1 = \frac{(w/2)}{\sqrt{z^2 + w^2/2}}$;

$B = \frac{\mu_0 I}{4\pi}\frac{w}{\sqrt{z^2 + (w^2/4)}\sqrt{z^2 + (w^2/2)}}$. To pick off the vertical component, multiply by $\sin\phi = \frac{(w/2)}{\sqrt{z^2 + (w/2)^2}}$; for all four sides, multiply by 4: $\boxed{\mathbf{B} = \frac{\mu_0 I}{2\pi}\frac{w^2}{(z^2 + w^2/4)\sqrt{z^2 + w^2/2}}\,\hat{\mathbf{z}}.}$ For $z \gg w$, $\mathbf{B} \approx \frac{\mu_0 I w^2}{2\pi z^3}\,\hat{\mathbf{z}}$. The field of a dipole $\boxed{m = Iw^2,}$ for points on the z axis (Eq. 5.86, with $r \to z$, $\hat{\mathbf{r}} \to \hat{\mathbf{z}}$, $\theta = 0$) is $\mathbf{B} = \frac{\mu_0}{2\pi}\frac{m}{z^3}\,\hat{\mathbf{z}}$.



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9. *[G 5.40] A plane wire loop of irregular shape is situated so that part of it is in a uniform magnetic field* **B** *(in Fig. the field occupies the shaded region, and points perpendicular to the plane of the loop). The loop carries a current I. Show that the net magnetic force on the loop is $F = IBw$, where w is the chord subtended. Generalize this result to the case where the magnetic field region itself has an irregular shape. What is the direction of the force?*

From Eq. 5.17, $\mathbf{F} = I\int(d\mathbf{l} \times \mathbf{B})$. But **B** is constant, in this case, so it comes outside the integral: $\mathbf{F} = I\left(\int d\mathbf{l}\right) \times \mathbf{B}$, and $\int d\mathbf{l} = \mathbf{w}$, the vector displacement from the point at which the wire first enters the field to the point where it leaves. Since **w** and **B** are perpendicular, $F = IBw$, and **F** is perpendicular to **w**.

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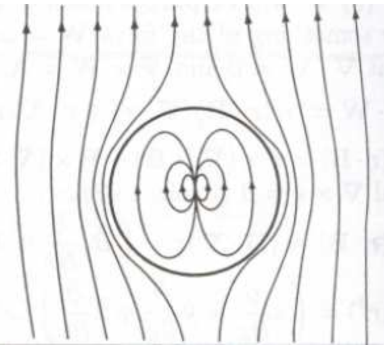
10. [**G 5.55**]A magnetic dipole$\mathbf{m} = -m_0\hat{\mathbf{z}}$ is situated at the origin, in an otherwise uniform magnetic field $\mathbf{B} = B_0\hat{\mathbf{z}}$. Show that there exists a spherical surface, centered at the origin, through which no magnetic field lines pass. Find the radius of this sphere and sketch the field lines, inside and out.

Problem 5.55



From Eq. 5.86, $\mathbf{B}_{\text{tot}} = B_0\,\hat{\mathbf{z}} - \frac{\mu_0 m_0}{4\pi r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})$. Therefore $\mathbf{B}\cdot\hat{\mathbf{r}} = B_0(\hat{\mathbf{z}}\cdot\hat{\mathbf{r}}) - \frac{\mu_0 m_0}{4\pi r^3}2\cos\theta = \left(B_0 - \frac{\mu_0 m_0}{2\pi r^3}\right)\cos\theta$. This is zero, for all θ, when $r = R$, given by $B_0 = \frac{\mu_0 m_0}{2\pi R^3}$, or $\boxed{R = \left(\frac{\mu_0 m_0}{2\pi B_0}\right)^{1/3}}$. Evidently no field lines cross this sphere.