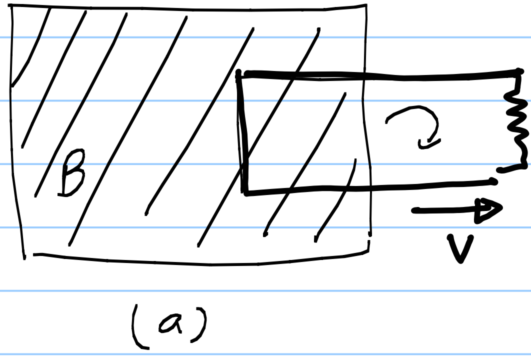
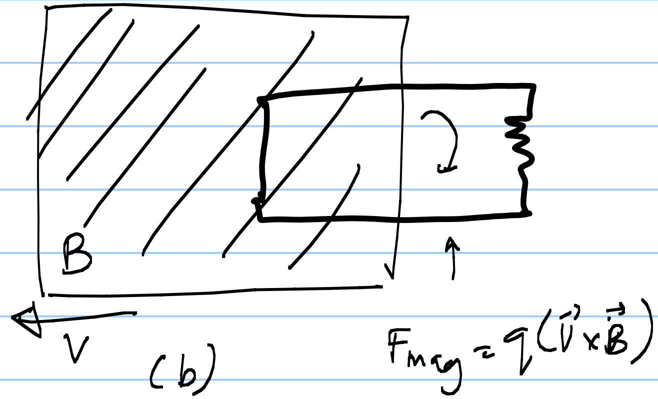
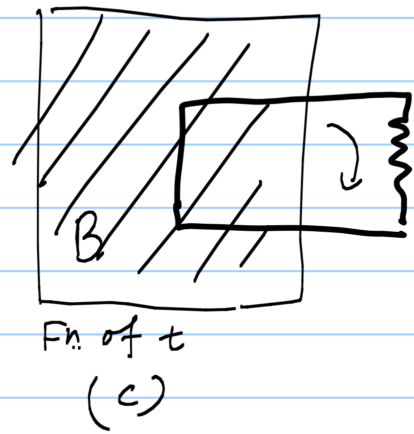
Faraday's Law

Note Title 4/13/2009







Faraday Law : Changing Magnetic Field induces Electric field.

Faraday's Law: In region of space, $\exists$ $B(t)$, induces electric field in such a way that for any simple closed loop C, the induced emf is given by

$$\oint_C \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int_S \vec{B} \cdot d\vec{s} \quad , \quad C \text{ encircles } S$$

Differential form:

$$\int_S \vec{\nabla} \times \vec{E} \cdot d\vec{s} = - \int_S \frac{\partial}{\partial t} B \cdot d\vec{s}$$

$$\int_S \left(\vec{\nabla} \times \vec{E} + \frac{\partial}{\partial t} \bar{B} \right) \cdot d\vec{s} = 0 \qquad \left\{ \begin{array}{l} \text{true for} \\ \text{any bounded} \\ \text{surface} \end{array} \right.$$

$$\Rightarrow \quad \boxed{\nabla \times \vec{E} = - \frac{\partial B}{\partial t}}$$

Compare (a) and (b)

Motional emf $\quad \mathcal{E}_M = -\dfrac{d\phi}{dt}$

induced emf $\quad \mathcal{E}_{\hat{e}} = -\dfrac{d\phi}{dt}$

$\Big\}$ Flux Rule

Motional emf $\Longleftrightarrow$ induced emf

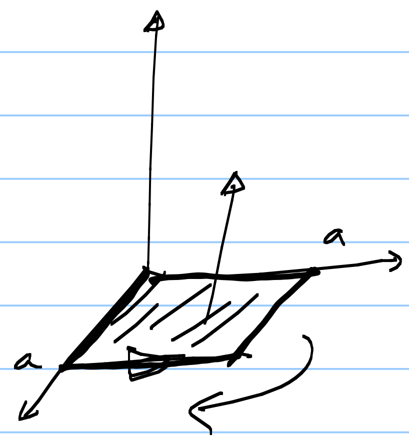
$\Rightarrow$ Sp. Theory of relativity

Example: Rectangular loop in XY plane

$B(y,t) = k\, y^3 t^2\, \hat{z} \qquad ds = dx\,dy\, \hat{z}$

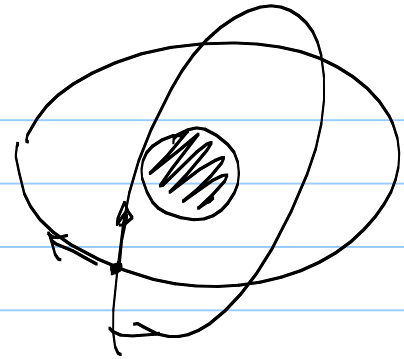
$\phi(t) = \int B \cdot ds = k t^2 \int_0^a \int_0^a dx\,dy\; y^3$

$= k t^2 \dfrac{a^5}{4}$



$$\mathcal{E}_{induced} = -\frac{d\Phi}{dt} = \frac{-ka^5 t}{2}$$

Ex



Faraday Law : induced Electric field

$$\nabla \times E_i = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \cdot E_i = 0$$

$$\nabla \times B = \mu_0 J$$

$$\nabla \cdot B = 0$$

$$\nabla \times E_i \neq 0$$

$$\nabla \times E_e = 0$$

$$\nabla \cdot E_i = 0$$

Ex $B(t) = \mu_0 n I(t)$ $\quad r < R$

$= 0$ $\quad r > R$

Use symmetry arg $\quad \vec{E} = E \hat{\phi}$

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\left(\mu_0 n I \cdot \pi s^2\right)$$

$$\Rightarrow 2\pi s E = -\mu_0 n \pi s^2 \, dI/dt$$

$$\vec{E} = -\frac{\mu_0 n s}{2} \frac{dI}{dt} \hat{\phi} \qquad s < R$$

$$\vec{E} = -\frac{\mu_0 n R^2}{2s} \frac{dI}{dt} \hat{\phi} \qquad s > R$$

Quasistatic Approximation:

Current not steady $\rightarrow$ BS Law approximate

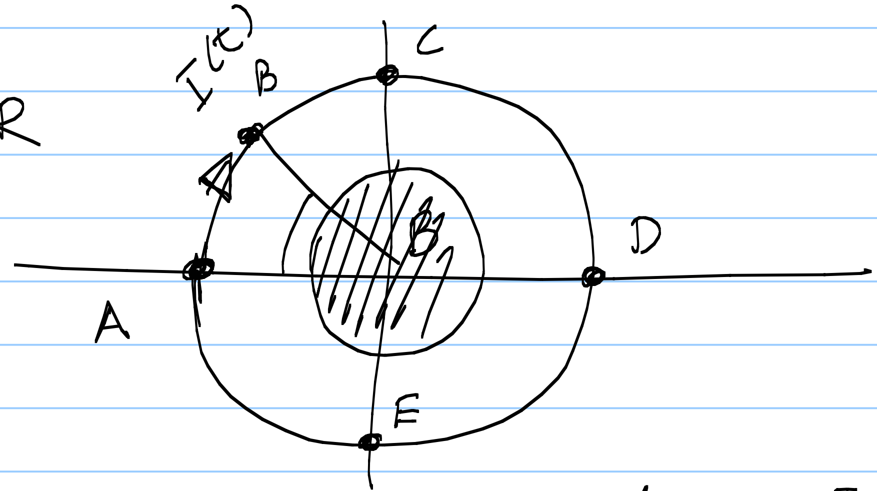
if s/c is not large

if $I(t)$ is slowly varying.

Ex.

wire: Resistance R

Perimeter L



$B(t)$ in shaded region.

At t:

$\mathcal{E}$

$V_{AB} = I \cdot \frac{R}{8}$ $\quad V_{AC} = \frac{IR}{4}$ $\quad V_{AD} = \frac{IR}{2}$

$V_{AE} = ?$ $\quad \frac{3IR}{4}$? $\quad V_{AA} = ?$

or $-\frac{IR}{4}$? ✓

Puzzle!

Mutual Inductance

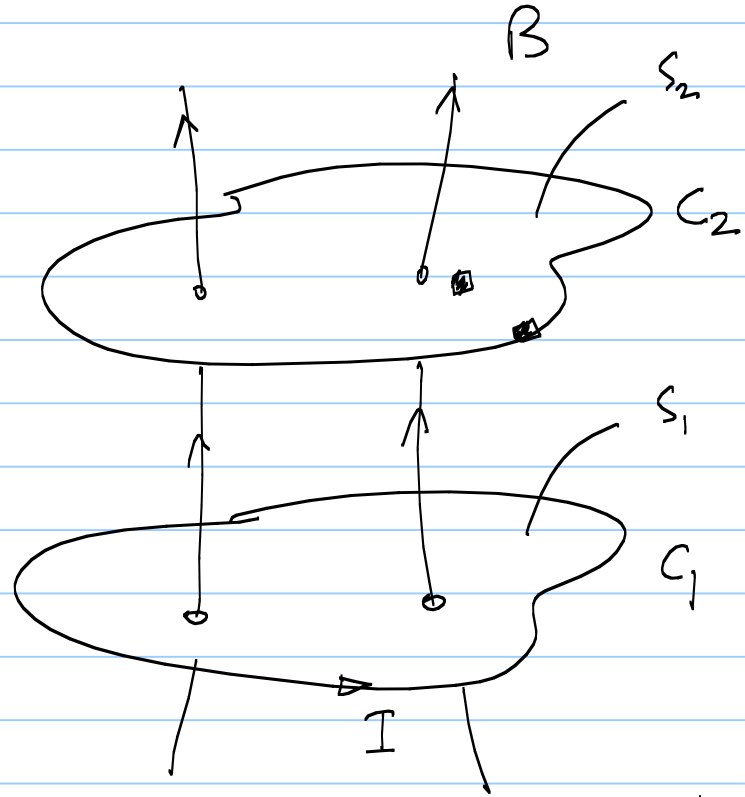
If I flows thro' C_1

$\Rightarrow$ Magnetic field B_1

$$\phi_2 = \int_{S_2} B_1 \cdot ds_2$$

$$B_1(r_2) = \frac{\mu_0 I}{4\pi} \int \frac{\vec{dl} \times (r_2 - r_1)}{|r_2 - r_1|^3}$$

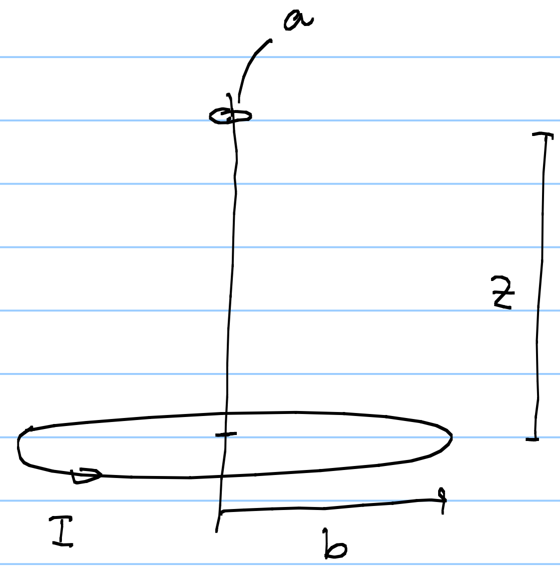
$$B_1 \propto I$$



C_1 and C_2 are stationary
fixed

$$\Phi_2 = \underbrace{M_{21}} I \qquad \Phi_1 = M_{12} I$$

Mutual inductance.



Ex $a << b$

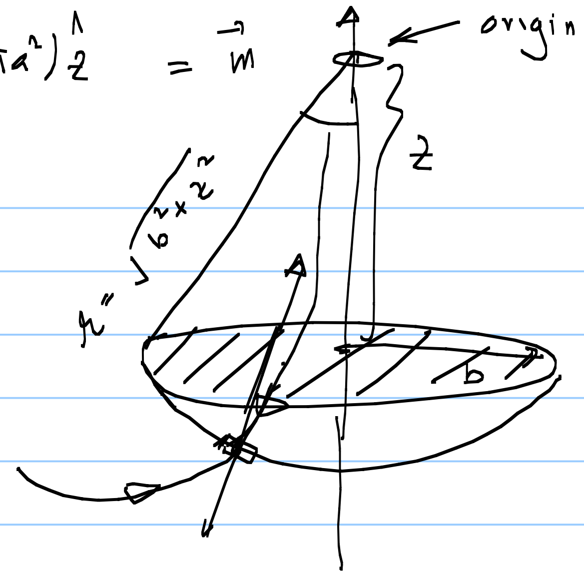
(a) $\vec{B} = \frac{\mu_0 I}{2} \frac{b^2}{(b^2+z^2)^{3/2}}$ ✓

$$\Phi = \frac{\mu_0 I}{2} \frac{b^2}{(b^2+z^2)^{3/2}} \underline{(\pi a^2)}$$

$$M_{ab} = \frac{\mu_0 \pi a^2 b^2}{2(b^2+z^2)^{3/2}}$$

(b)

$$\vec{B} = \frac{\mu_0 m}{4\pi r^3}\left(\hat{r}\,(2\cos\theta) + \hat{\theta}\sin\theta\right) \qquad I(\pi a^2)\hat{z} = \vec{m}$$



$$\vec{ds} = 2\pi r^2 \sin\theta\, d\theta\,(-\hat{r})$$

$$\sin\theta_0 = b/r$$

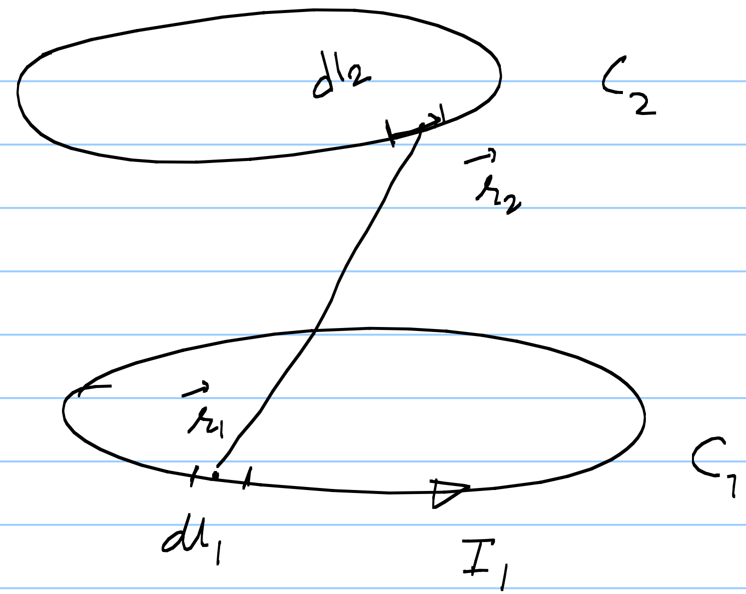
$$\phi_b = \int \vec{B}\cdot\vec{ds} = -\frac{\mu_0 m}{4\pi r^3}\int_{\pi-\theta_1}^{\pi} 2\pi r^2 \sin\theta\, d\theta \times 2\cos\theta$$

$$= \frac{\mu_0 \pi}{2}\,\frac{a^2 b^2}{(b^2+z^2)^{3/2}}\, I$$

$$M_{ba} = \frac{\mu_0 \pi}{2}\,\frac{a^2 b^2}{(b^2+z^2)^{3/2}}$$

Possibly $M_{ba} = M_{ab}$ always?

Neumann Formula



$$\phi_2 = \int_{S_2} B \cdot dS_2$$

$$= \int_{S_2} (\nabla \times A) \cdot dS_2 = \oint_{C_2} \bar{A}(r_2) \cdot d\bar{l}_2$$

$$A(r_2) = \frac{\mu_0}{4\pi} \oint_{C_1} \frac{I_1 \, d\vec{l}_1}{|r_2 - r_1|}$$

$$\phi_2 = \frac{\mu_0 I}{4\pi} \oint_{C_1} \oint_{C_2} \frac{d\vec{l}_1 \cdot d\vec{l}_2}{|r_2 - r_1|}$$

$$M_{21} = \frac{\mu_0}{4\pi} \oint_{C_1} \oint_{C_2} \frac{dl_1 \cdot dl_2}{|r_2 - r_1|}$$

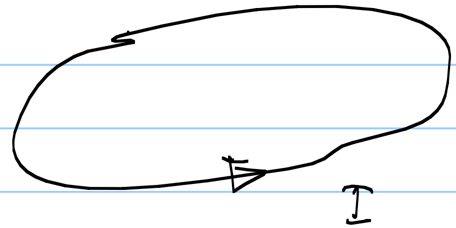
$\Rightarrow \quad M_{21} = M_{12} = M$

$\Rightarrow \quad \mathcal{E}_2 = -M \dfrac{dI}{dt}$

$\Rightarrow$ Geometrical

Self Inductance

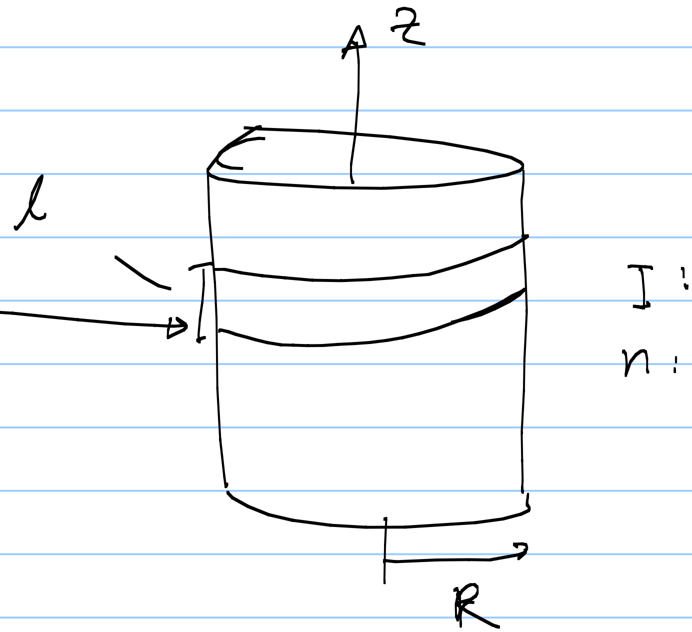
$$L = \frac{\phi}{I}$$ at instant



<u>Ex</u> $\vec{B} = \mu_0 n I \hat{z}$

Flux thro' single loop

$= \mu_0 n I \cdot \pi R^2$



Flux thro' length l

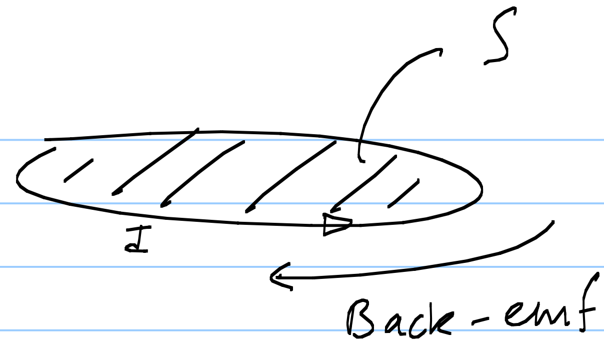
$= (\mu_0 n I \pi R^2) n l$

Self Inductance/unit length $= \mu_0 n^2 \pi R^2$

Energy

At $t=0$ $\quad I=0$

t $\quad I$



Power need to drive changes

$$\frac{dW}{dt} = -\mathcal{E} I = +L I \frac{dI}{dt}$$

$$\frac{1}{2}\int \rho V \, d\tau$$

$$W = \frac{1}{2} L I^2 = \frac{1}{2} \phi I$$

$$= \frac{1}{2} I \int B \cdot dS$$

$$= \frac{1}{2} I \int (\nabla \times A) \cdot ds = \frac{1}{2} \oint_c (\vec{A} \cdot \vec{I}) \, dl$$

Generalize $\quad W = \frac{1}{2} \int_v (\vec{A} \cdot \vec{J}) \, dv$

$\mu_0 J = \nabla \times B$

$$= \frac{1}{2\mu_0} \int_v \left[\vec{A} \cdot (\nabla \times B) \right] dv$$

$$= \frac{1}{2\mu_0} \int_v \left[B \cdot (\nabla \times A) - \nabla \cdot (A \times B) \right] dv$$

$$\nabla \cdot (\vec{A} \times B) = B \cdot (\nabla \times A) - A \cdot (\nabla \times B)$$

$$= \frac{1}{2\mu_0} \int_v B^2 dv - \underbrace{\oint_S (A \times B) \cdot d\vec{s}}_{\Downarrow\, 0}$$

if S is at infinity

$$W_{mag} = \frac{1}{2\mu_0} \int_{\text{entire space}} B^2 dv$$

$$W_{elec} = \frac{\epsilon_0}{2} \int E^2 dv$$

Maxwell's Equations

4 Eq.

$$\nabla \cdot E = \frac{1}{\epsilon_0} \rho$$ Gauss Law

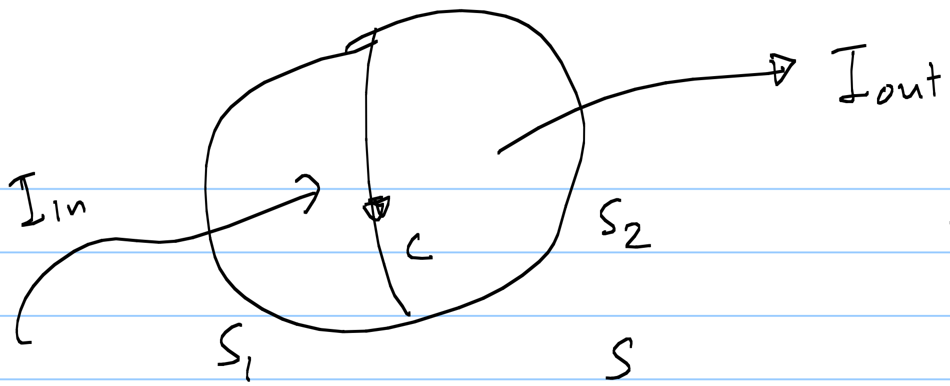
$$\nabla \cdot B = 0$$ Law of no monopoles

$$\nabla \times E = -\frac{\partial B}{\partial t}$$ Faraday Law

$$\nabla \times B = \mu_0 J$$ Ampere Law

For ES $\nabla \times E = 0$ replaced by Faradays

For MS $\nabla \cdot J = 0$ $\nabla \cdot J = -\frac{\partial \rho}{\partial t} \neq 0$



$\nabla \cdot J = 0$

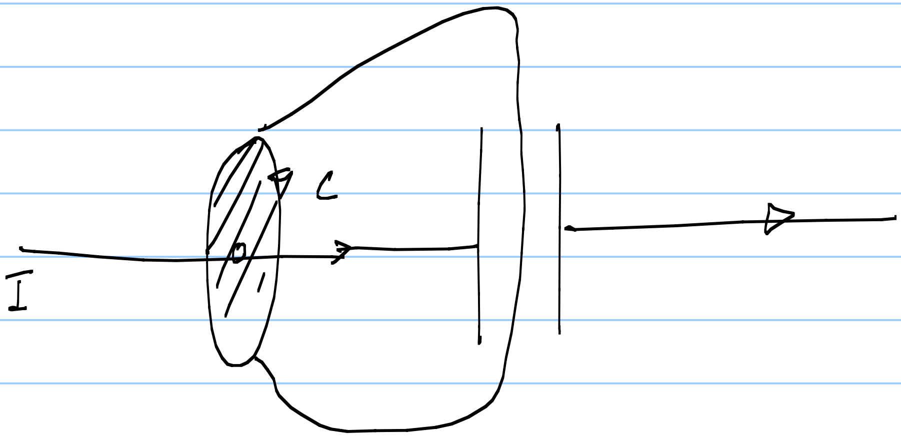
$\Rightarrow \quad I_{in} = I_{out}$

if $\quad \nabla \cdot J \neq 0 \qquad I_{in} \neq I_{out}$

Ampere's $\qquad \oint_C \bar{B} \cdot \bar{dl} = \mu_0 I_{enc} = \mu_0 I_{in}$

$$= \mu_0 I_{out}$$

Ex



$$\nabla \cdot (\nabla \times E) = \nabla \cdot \left(-\frac{\partial B}{\partial t}\right) = -\frac{\partial}{\partial t}(\nabla \cdot B)$$

$$0 \overset{\checkmark}{=} 0$$

$$\nabla \cdot (\nabla \times B) = \nabla \cdot (\mu_0 J) = \mu_0 \nabla \cdot J$$

$$0 \overset{?}{=} \text{nonzero}$$

Cont.

$$\nabla \cdot J = -\frac{\partial}{\partial t}\rho = -\frac{\partial}{\partial t}(\epsilon_0 \nabla \cdot E)$$

$$\Rightarrow \quad \nabla \cdot \big(\underbrace{J + \epsilon_0 \frac{\partial E}{\partial t}}\big) = 0$$

Maxwell

$$\nabla \times B = \mu_0 J + \mu_0 \epsilon_0 \frac{\partial E}{\partial t}$$

1861

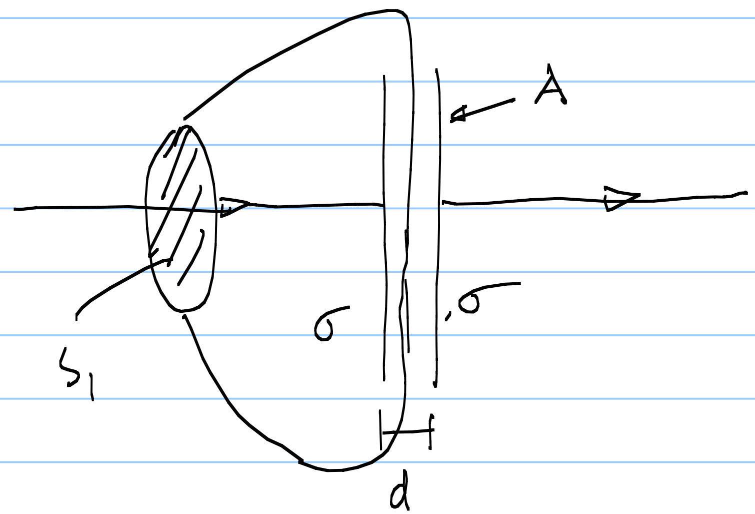
$$\epsilon_0 \frac{\partial E}{\partial t} = J_d = \text{Displacement Current}$$

$$\nabla \times E \propto -\frac{\partial B}{\partial t}$$

$$\nabla \times B \propto \frac{1}{c^2}\frac{\partial E}{\partial t} \simeq 10^{-16}\frac{\partial E}{\partial t} \qquad (1877)$$

$$\boxed{\text{Changing Electric fields induce Magnetic}}$$

Ex



$E = \dfrac{\sigma}{\epsilon_0}$ inside capacitor

$= 0$ outsid Capacitor

Thro' S_1 : $E = 0$ $\qquad I_{enc} = I$

Thro' S_2 : $J = 0$ $\qquad \epsilon_0 \dfrac{\partial E}{\partial t} = \dfrac{\partial \sigma}{\partial t} = I$

Final Form

$$\nabla \cdot E = +\rho/\epsilon_0$$
$$\nabla \cdot B = 0$$
$$\nabla \times E + \partial B/\partial t = 0$$
$$\nabla \times B - \mu_0 \epsilon_0 \partial E/\partial t = \mu_0 J$$

$$F_{Lorentz} = q(\vec{E} + \vec{v} \times \vec{B})$$

Fields $\Longleftarrow$ Charges
Currents

Charge Cons.

Wave Equation: In vacc $\rho = 0$

$J = 0$

$\nabla \times E = -\partial B/\partial t$ $\qquad$ $\nabla \cdot E = 0$

$\nabla \times B = \mu_0 \epsilon_0 \cdot \partial E/\partial t$ $\qquad$ $\nabla \cdot B = 0$

Curl

$$\nabla \times (\nabla \times E) = \nabla(\underbrace{\nabla \cdot E}) - \nabla^2 E = -\partial/\partial t \, (\nabla \times B)$$

$$-\nabla^2 E = -\mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} E$$

$$\frac{\partial^2}{\partial x^2} E = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} E \qquad \text{(Wave Eqn)}$$

Speed of wave $c = \dfrac{1}{\sqrt{\mu_0 \epsilon_0}}$

1710 $\qquad c \simeq 3.02 \times 10^{+8}$ m/s $\quad$ } Astronomy

1861 $\quad c \simeq 3.00 \times 10^8$ m/s

$1/\sqrt{\mu_0 \epsilon_0} =$

laboratory constants

light $\longleftrightarrow$ EM wave