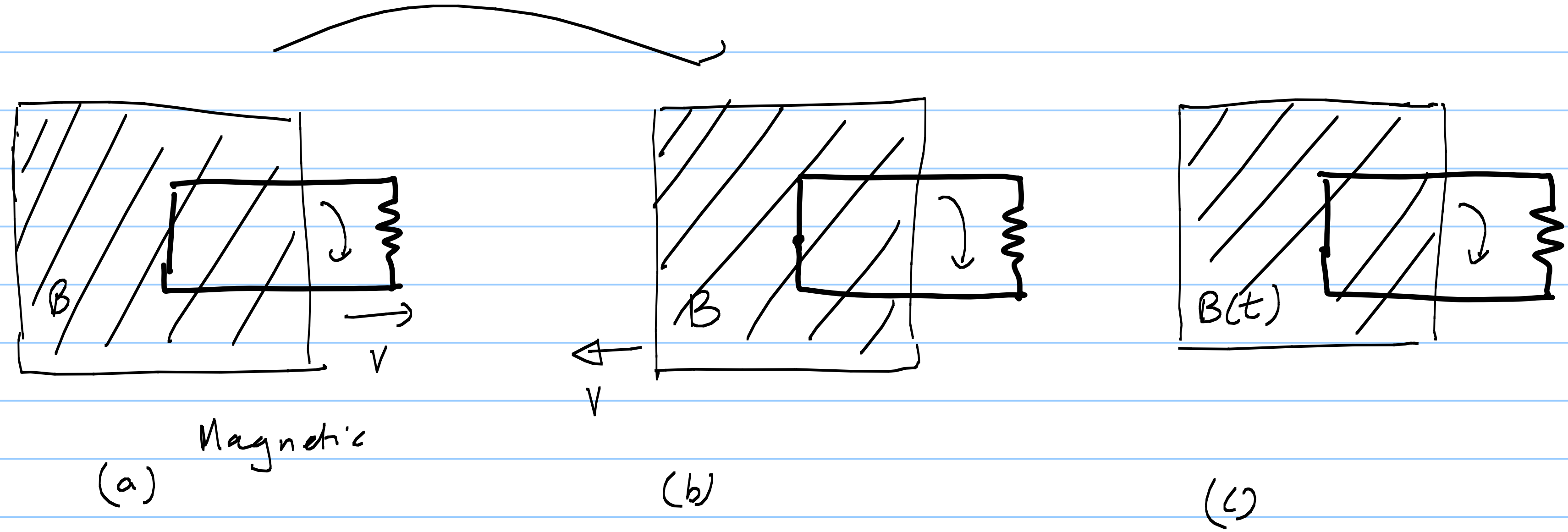
Faraday's Law (E)

Note Title 4/13/2009



Magnetic force depends on frame of ref..

Faraday's Law : Changing Magnetic fields induces Electric field

Faraday's Law: If in a region of space, MF $B(t, \vec{r})$, then it induces electric field s.t. in <u>any closed</u> simple loop C,

$$\oint_C \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int_S \vec{B} \cdot ds$$

C is boundary of S

$\boxed{\text{Experimental Law}}$

Differential Form:

$$\int_S (\bar{\nabla} \times \bar{E}) \cdot d\bar{s} = -\int_S \frac{\partial}{\partial t} B \cdot ds$$

$$\Downarrow$$

$$\Rightarrow \int_S \left(\bar{\nabla} \times \bar{E} + \frac{\partial}{\partial t} \bar{B}\right) \cdot ds = 0$$

true for all bounded S

$$\Rightarrow \boxed{\nabla \times \vec{E} = -\frac{\partial B}{\partial t}}$$

Compare this

(a) Motional Emf $= \oint_c \bar{f}_{mag} \cdot d\bar{l} = -\dfrac{d\phi}{dt}$

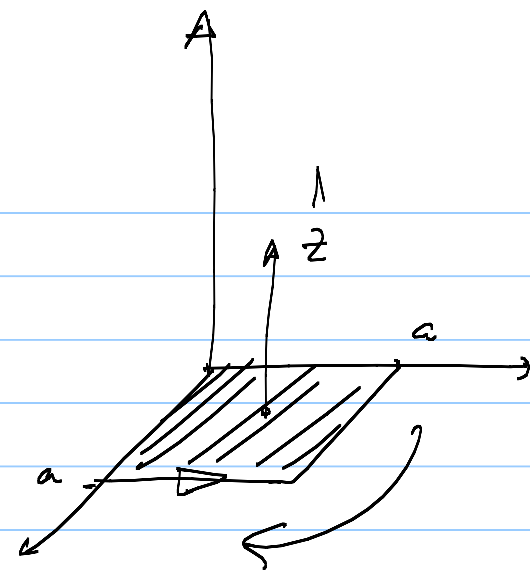
(b) Induced Emf $= \oint \vec{E} \cdot d\bar{l} = -\dfrac{d\phi}{dt}$

Motional emf $\Longleftrightarrow$ Induced Emf

$\hookrightarrow$ Sp Theory of Relativity

Ex Square loop of side a

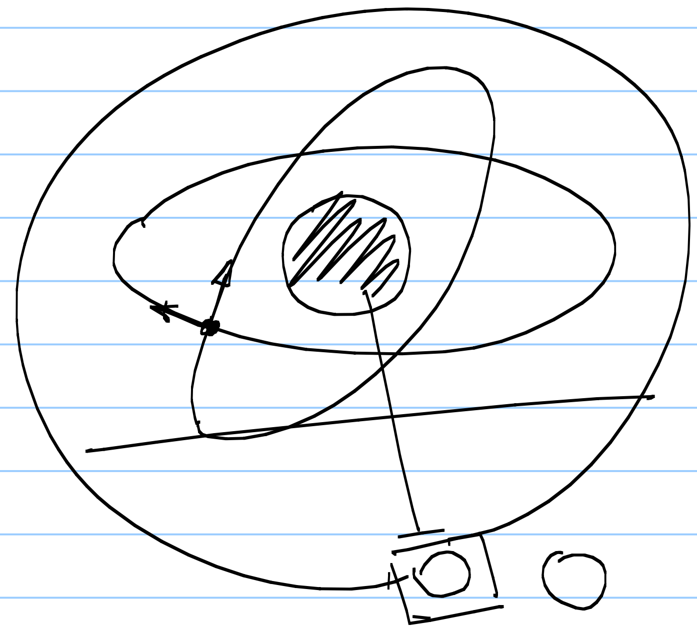
$$B(y,t) = k y^3 t^2 \hat{z}$$



$$\phi = \int \vec{B} \cdot d\vec{s} \qquad d\vec{s} = dx\,dy\,\hat{z}$$

$$= k t^2 \frac{a^5}{4}$$

$$\mathcal{E} = -\frac{d\phi}{dt} = \frac{-k a^5 t}{2}$$



$$\nabla \times E_i \neq 0$$

$$\nabla \times E_q = 0$$

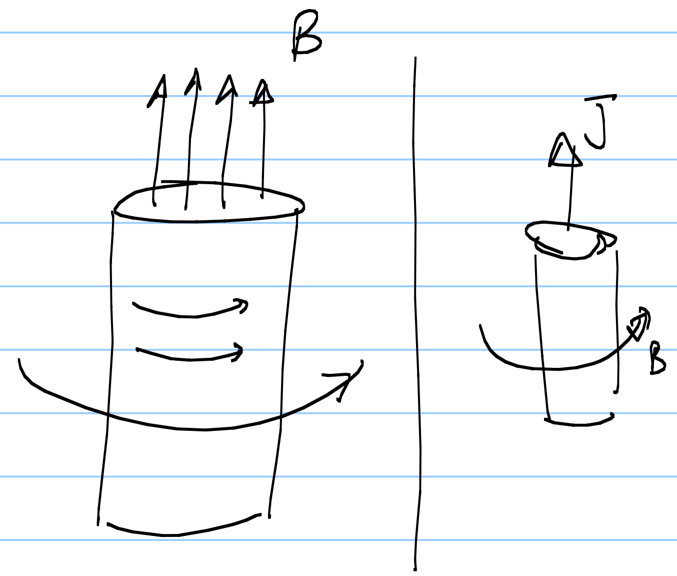
For induced EF

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \cdot E = 0$$

$$\bar{\nabla} \times \bar{B} = \mu_0 J$$

$$\nabla \cdot B = 0$$



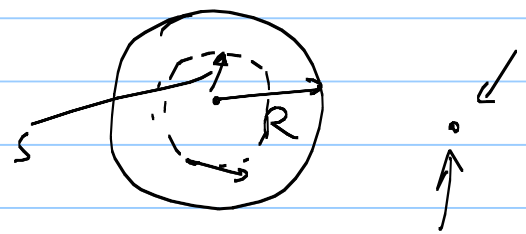
Ex long Solenoid

$$\vec{B}(t) = \mu_0 n I(t) \hat{z}$$

Direction of E is $\hat{\phi}$

$$\oint \bar{E} \cdot d\bar{l} = 2\pi s E = -\frac{d\phi}{dt}$$

$$= -\mu_0 n (\pi s^2) \frac{dI}{dt}$$



$$\vec{E} = -\frac{\mu_0 n s}{2} \frac{dI}{dt} \hat{\phi} \qquad s < R$$

$$= -\frac{\mu_0 n R^2}{2s} \frac{dI}{dt} \hat{\phi} \qquad s > R$$

Quasistatic Approximation

$\rightarrow \quad \tau = \dfrac{S}{c}$ small enough

$\rightarrow \quad I(t)$: varying slowly

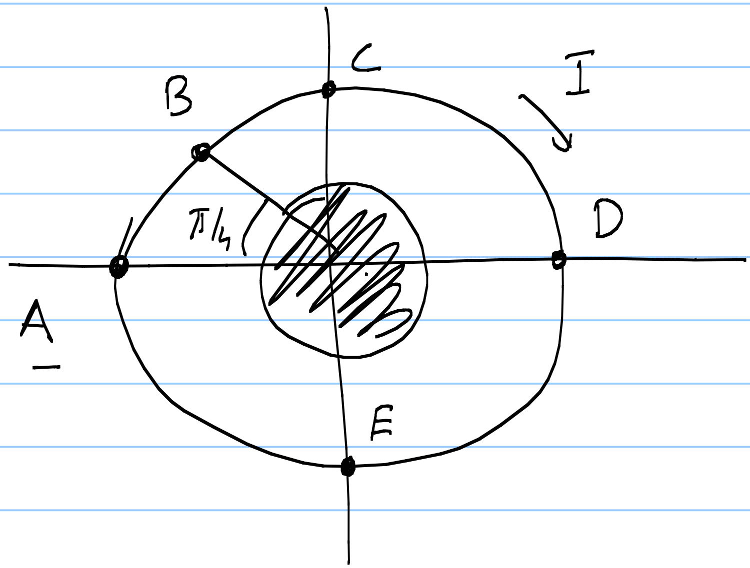
Ex Circular wire

R : Resistance

L : Perimeter

At t : $\mathcal{E}$: emf

I : Current



B(t) : in shaded region.

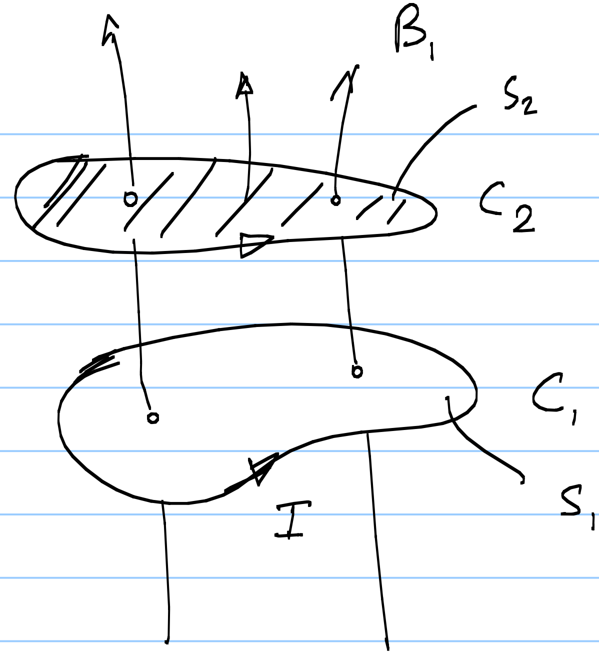
Puzzle!

$V_{AB} = \dfrac{IR}{8} \qquad V_{AC} = \dfrac{IR}{4} \qquad V_{AD} = \dfrac{IR}{2}$

$V_{AE} = ? \quad \dfrac{3IR}{4}$ or $-\dfrac{IR}{4}$

Mutual Inductance

C_1 and C_2 fixed stationary loops



Flux thro' C_2

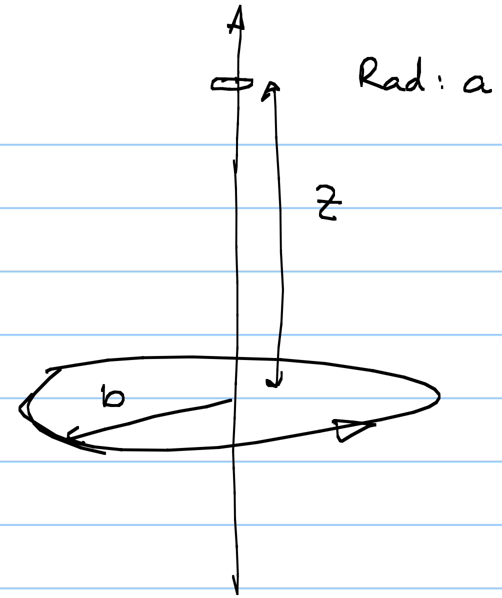
$$= \int_{S_2} \bar{B}_1 \cdot d\bar{s}_2$$

$B_1 \propto I$

$\phi_2 = M_{21} I$ M_{21} : Mutual Inductance

Ex

(a) $$\vec{B}_b(z) = \frac{\mu_0 I}{2} \frac{b^2}{(b^2+z^2)^{3/2}}$$



$$\phi_a = \left[\frac{\mu\pi}{2} \frac{a^2 b^2}{(b^2+z^2)^{3/2}}\right] I$$

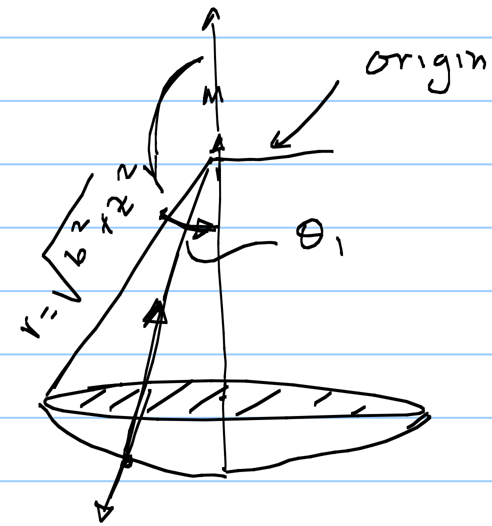
$$M_{ba} = \frac{\mu\pi}{2} \frac{a^2 b^2}{(b^2+z^2)^{3/2}}$$

(b) $$\vec{m} = I(\pi a^2)\hat{z}$$

$$\sin(\theta_1) = \frac{b}{r}$$

$$d\vec{s} = 2\pi r^2 \sin\theta \, d\theta(-\hat{r})$$

$$\vec{B} = \frac{\mu_0 m}{4\pi r^3}\left[\hat{r}(2\cos\theta) + \hat{\theta}\sin\theta\right]$$

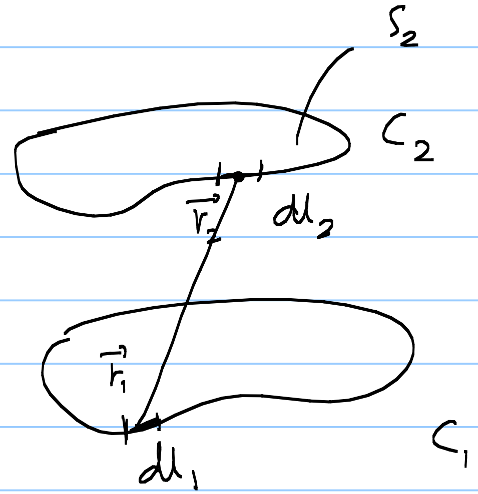


$$\phi = \int \vec{B} \cdot d\vec{s} = \frac{-\mu_0 m}{4\pi r^3} \int_{\pi - \theta_1}^{\pi} 2\cos\theta \cdot 2\pi r^2 \sin\theta \, d\theta$$

$$= \left[\frac{\mu_0 \pi}{2} \frac{a^2 b^2}{(b^2 + z^2)^{3/2}} \right] I$$

$$M_{ab} = M_{ba}$$

Neumann Formula



$$\phi_2 = \int_{S_2} \vec{B}_1 \cdot d\vec{s}_2 = \int_{S_2} (\bar{\nabla} \times \bar{A}_1) \cdot d\vec{s}_2$$

$$= \oint_{C_2} A_1(r_2) \cdot d\vec{l}_2$$

$$A_1(r_2) = \frac{\mu_0 I}{4\pi} \oint_{C_1} \frac{d\vec{l}_1}{|r_2 - r_1|}$$

$$\phi_2 = \underbrace{\left[\frac{\mu_0}{4\pi} \oint_{C_1} \oint_{C_2} \frac{d\bar{l}_1 \cdot d\bar{l}_2}{|r_2 - r_1|} \right]}_{M_{21}} I$$

$$\Rightarrow \quad M_{12} = M_{21}$$

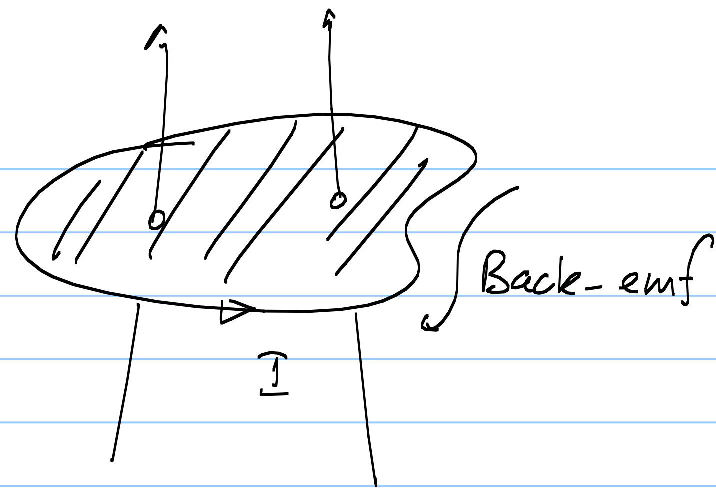
$\Rightarrow$ M is geometrical property

$$\Rightarrow \quad \mathcal{E}_2 = -M \frac{dI}{dt}$$

Self Inductance

$$L = \phi / I$$

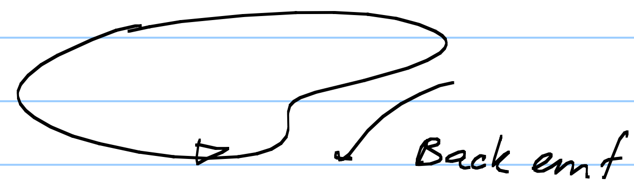
$$\mathcal{E} = -L \frac{dI}{dt}$$



Energy in Magnetic fields

At $t=0$ $I=0$

At t I



Power needed to drive changes

$$\frac{dW}{dt} = -\mathcal{E} I$$

(Work done by ext agency)

$$= L \frac{dI}{dt} I$$

Energy $\quad W = \frac{1}{2} L I^2$

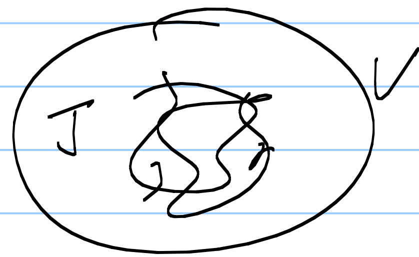
$$= \frac{1}{2} \phi I$$

ϕ : flux

$$= \frac{1}{2} I \int \vec{B} \cdot d\vec{S} = \frac{1}{2} I \int (\nabla \times A) \cdot ds$$

$$= \frac{1}{2} \oint_c (\bar{A} \cdot \bar{I})\, dl$$

$$W = \frac{1}{2} \int_V (\bar{A} \cdot \bar{J})\, dv$$



$$W_{elec} = \frac{1}{2} \int_V \rho V\, dv$$

Electric Potential.

$$W = \frac{1}{2\mu_0}\int_V \left(\vec{A}\cdot(\nabla\times B)\right) dv \qquad J = \frac{1}{\mu_0}(\nabla\times B)$$

$$\nabla\cdot(A\times B) = B\cdot(\nabla\times A) - A\cdot(\nabla\times B)$$

$$W = \frac{1}{2\mu_0}\int \left[B\cdot(\nabla\times A) - \nabla\cdot(A\times B)\right] dv$$

$$= \frac{1}{2\mu_0}\int_V |B|^2 dv - \oint_S (A\times B)\cdot d\bar{s}$$

$\Rightarrow 0$ if S is at infinity

$$= \frac{1}{2\mu_0}\int_{\text{entire space}} |B|^2 dv$$

$$W_{elc} = \frac{\epsilon_0}{2}\int_{\text{entire space}} |E|^2 dv$$

Maxwell's Equations

$$\nabla \cdot E = \frac{\rho}{\epsilon_0}$$ Gauss Law

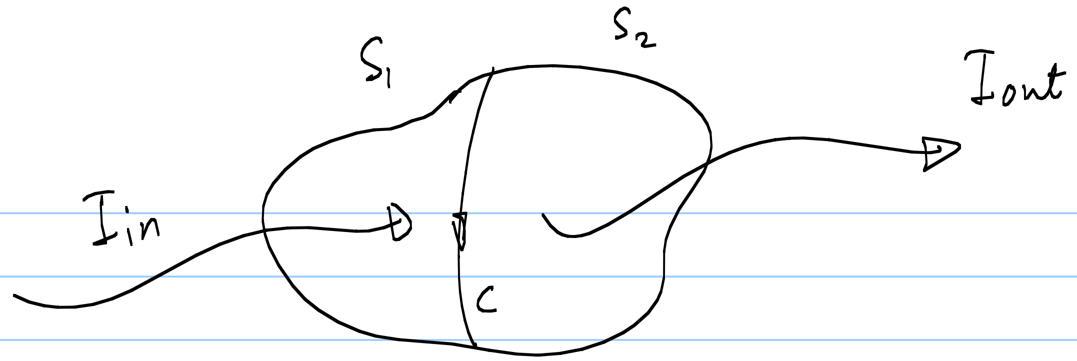
$$\nabla \cdot B = 0$$ Law of no monopoles

$\nabla \times E = 0 \longrightarrow$ $$\nabla \times E = -\frac{\partial B}{\partial t}$$ Faraday's Law

$$\nabla \times B = \mu_0 J$$

Steady currents $\nabla \cdot J = 0$ $\quad \nabla \cdot J = -\frac{\partial \rho}{\partial t} = 0$

when $-\frac{\partial \rho}{\partial t} = \nabla \cdot J \neq 0$

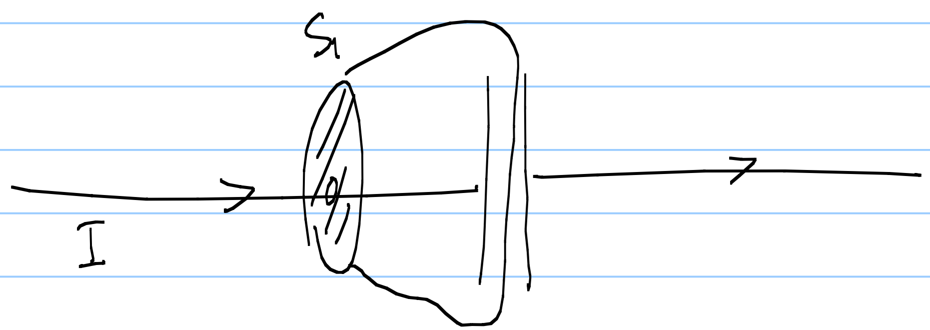


$\nabla \cdot J = 0 \quad \Rightarrow \quad I_{in} = I_{out}$

$\nabla \cdot J \neq 0 \quad \Rightarrow \quad I_{in} \neq I_{out}$

$$\left.\begin{aligned} \oint_C \bar{B} \cdot d\bar{l} = \mu_0 I_{enc} &= \mu_0 I_{in} \\ &= \mu_0 I_{out} \end{aligned}\right\} \quad Q?$$

Ex



Eq: Inconsistent

$$\nabla\cdot(\nabla\times E) = \nabla\cdot\left(-\frac{\partial B}{\partial t}\right) = -\frac{\partial}{\partial t}(\nabla\cdot B)$$

$$0 \overset{\checkmark}{=} 0$$

$$\nabla\cdot(\nabla\times B) = \mu_0(\nabla\cdot J)$$

0 = need not be zero

Cont. Eq

$$\nabla\cdot J = -\frac{\partial\rho}{\partial t} = -\frac{\partial}{\partial t}(\epsilon_0\nabla\cdot E)$$

$$\nabla\cdot\left(J + \epsilon_0\frac{\partial E}{\partial t}\right) = 0 \qquad \epsilon_0\frac{\partial E}{\partial t} = J_d$$

= Displacement Current

1861

Fourth: $\nabla\times B = \mu_0 J + \mu_1\epsilon_0\frac{\partial E}{\partial t}$ ✓

$$\nabla \times E \propto -\frac{\partial B}{\partial t}$$

$$\nabla \times B \propto \underbrace{\mu_0 \epsilon_0}_{} \frac{\partial E}{\partial t} \approx 10^{-16} \frac{\partial E}{\partial t}$$

1877

$$\nabla \cdot E = \frac{\rho}{\epsilon_0}$$

$$\nabla \cdot B = 0$$

$$\nabla \times E + \frac{\partial B}{\partial t} = 0$$

$$\nabla \times B - \mu_0 \epsilon_0 \frac{\partial E}{\partial t} = \mu_0 J$$

Charge Cons.

$$F_L = q(\bar{E} + \vec{v} \times \bar{B}) = m\ddot{r}$$

Wave Eq. $\rho = 0$ $J = 0$

$$\nabla \cdot E = 0 \qquad \nabla \times E = -\partial B / \partial t$$
$$\nabla \cdot B = 0 \qquad \nabla \times B = \mu_0 \epsilon_0 \, \partial E / \partial t$$

$$\nabla \times (\nabla \times E) = \nabla \underbrace{(\nabla \cdot E)}_{= 0} - \nabla^2 E = -\frac{\partial}{\partial t} (\nabla \times B)$$
$$= -\mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} E$$

in 1D $$\boxed{\frac{\partial^2}{\partial x^2} E = + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} E}$$

Speed of wave c

1710 Astronomical $\Rightarrow$ $c = 3.02 \times 10^8 \frac{m}{s}$

1861 "

$$\Rightarrow \quad c = 3.00 \times 10^{8} \frac{m}{s}$$