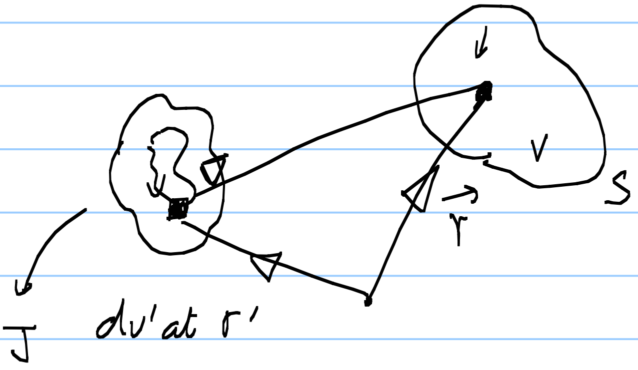
Magnetostatics (M)

Note Title 4/1/2009

(1) Divergence of B



$\vec{\nabla}$: gradient w.r.t $\vec{r}$

$\vec{\nabla}'$: gradient w.r.t $\vec{r}'$

$$B(\vec{r}) = \frac{\mu_0}{4\pi} \int dv' \, \frac{J(\vec{r}') \times (\vec{r} - \vec{r}')}{|r - r'|^3}$$

$$\nabla \cdot B(\vec{r}) = \frac{\mu_0}{4\pi} \int dv' \; \nabla \cdot \left[\frac{J(\vec{r}') \times (r - r')}{|r - r'|^3} \right]$$

$$\nabla \cdot [F \times G] = G \cdot (\nabla \times F) - F \cdot (\nabla \times G)$$

F is ind of $\vec{r}$ $\Rightarrow$ $\nabla \times F = 0$

$$= - F \cdot (\nabla \times G)$$

$$\left\{ \begin{array}{l} F(\vec{r}') = \bar{J}(r') \\ \vec{G}(r, r') = \dfrac{(r - r')}{|r - r'|^3} \end{array} \right.$$

$$\nabla \cdot B(\vec{r}) = \frac{-\mu_0}{4\pi} \int dv' \, J(r') \cdot \underbrace{\left(\nabla \times \frac{(r-r')}{|r-r'|^3} \right)}_{0}$$

$$\boxed{\nabla \cdot B = 0}$$

$$\int_V \nabla \cdot B \, dv = 0$$

Compare

$$\Rightarrow \boxed{\oint_S \vec{B} \cdot d\vec{s} = 0}$$

$$\oint_S E \cdot d\vec{s} = \frac{1}{\epsilon_0} \underbrace{Q_{net}}$$

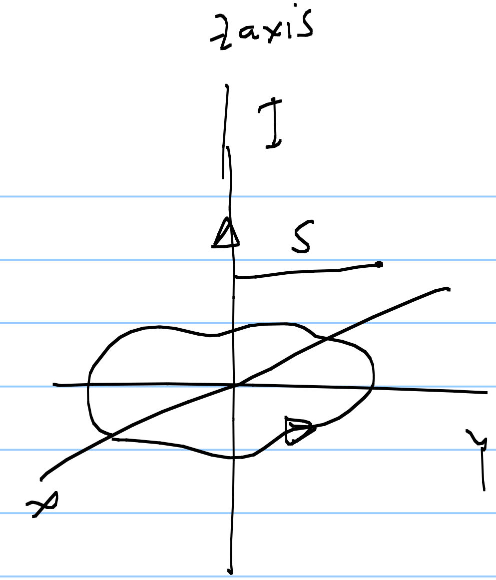
Magnetic monopoles don't Exist !

Magnetic field lines are endless

(ii)

Curl of $\vec{B}$

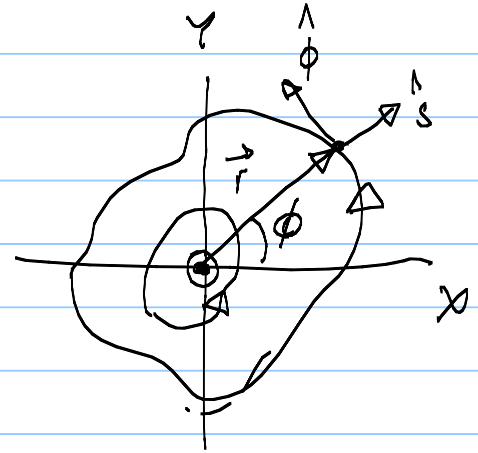
(a) Example: A long st. wire, I



$$\vec{B}(s) = \frac{\mu_0 I}{2\pi s}\,\hat{\phi}$$

A curve in XY plane

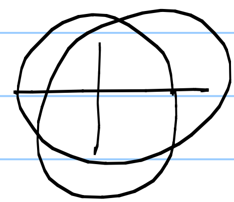
$$\underbrace{C: \quad \vec{r}(\phi) = (s(\phi), \phi, 0)}_{\phi:\, 0 \to 2\pi}$$ in cylindrical co-ordinates



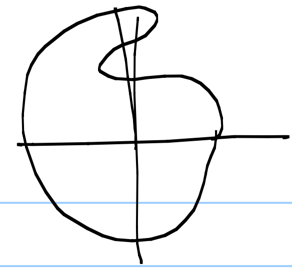
$$\vec{r}(\phi) = s(\phi)\,\hat{s}(\phi)$$

$$d\vec{r} = \frac{d\vec{r}}{d\phi}\,d\phi$$

$$= \left[\frac{ds}{d\phi}\,\hat{s} + s\,\frac{d\hat{s}}{d\phi}\right] d\phi +$$

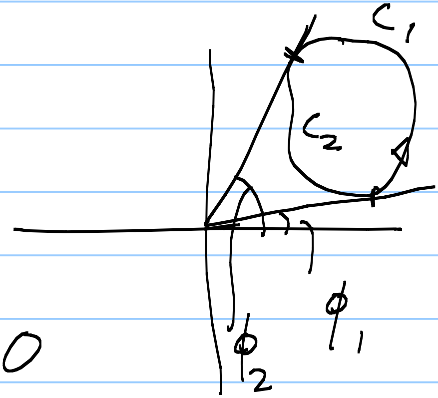


$$= \left[\frac{ds}{d\phi}\,\hat{s} + s\,\hat{\phi} \right] d\phi$$



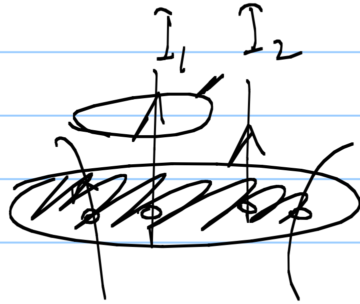
$$\oint_C B \cdot d\vec{r} = \int_0^{2\pi} \frac{\mu_0 I}{2\pi s}\, \hat{\phi} \cdot \left[\frac{ds}{d\phi}\,\hat{s} + s\,\hat{\phi} \right] d\phi$$

$$= \frac{\mu_0 I}{2\pi} \int_0^{2\pi} d\phi = \underline{\mu_0 I}$$



$$\oint_C B \cdot d\vec{r} = \int_{C_1} \vec{B} \cdot d\vec{r} + \int_{C_2} \vec{B} \cdot d\vec{r}$$

$$= \frac{\mu_0 I}{2\pi} \left[\int_{\phi_1}^{\phi_2} d\phi + \int_{\phi_2}^{\phi_1} d\phi \right] = \underline{0}$$



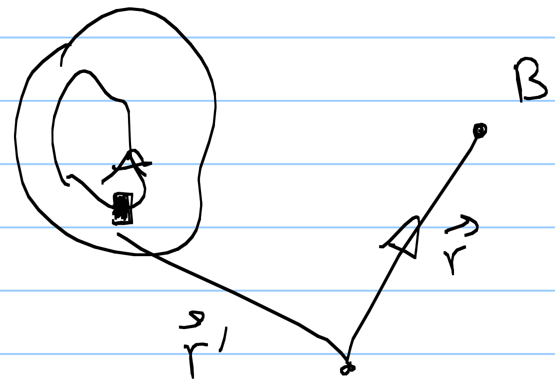
$$\oint_C \vec{B}\cdot d\vec{r} = \mu_0 I_{enclosed}$$

$$\int_S \nabla\times B\cdot ds = \mu_0 \int_S \vec{J}\cdot d\vec{s}$$

$$\boxed{\nabla\times B = \mu_0 J}$$

(b) $\vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$

$$\nabla\times B(\vec{r}) = \frac{\mu_0}{4\pi}\int dv'\,\nabla\times\left[\underbrace{J(\vec{r}\,')}_{A}\times\underbrace{\frac{\vec{\mathscr{r}}}{\mathscr{r}^3}}_{B}\right]$$



$$\nabla \times (A \times B) = \cancel{(B \cdot \nabla) A}^{0} - (A \cdot \nabla) B + A (\nabla \cdot B) - B \cancel{(\nabla \cdot A)}^{0}$$

A is ind of $\vec{r}$

$$= A(\nabla \cdot B) - (A \cdot \nabla) B$$

$$\nabla \times \left[J(\vec{r}') \times \frac{\vec{\mathscr{r}}}{\mathscr{r}^3} \right] = J \underbrace{\left(\nabla \cdot \frac{\vec{\mathscr{r}}}{\mathscr{r}^3} \right)}_{J\, 4\pi\, \delta(\vec{\mathscr{r}})} - (J \cdot \nabla) \frac{\vec{\mathscr{r}}}{\mathscr{r}^3}$$

$$\nabla \times B(\vec{r}) = \frac{\mu_0}{4\pi} \underline{\int dv' \, J(\vec{r}') \, 4\pi \, \delta(\vec{r} - \vec{r}')} + \underbrace{\text{Second term}}_{\substack{\shortparallel \\ 0}}$$

$$= \mu_0 J(\vec{r})$$

$$\underbrace{\frac{\mu_0}{4\pi} \oint_S \underline{\underline{J(\vec{r}')}} \frac{(x - x')}{\mathscr{r}^3} \cdot ds}_{\text{x Component of ST}}$$

$$\boxed{\nabla \times B = \mu_0 J}$$

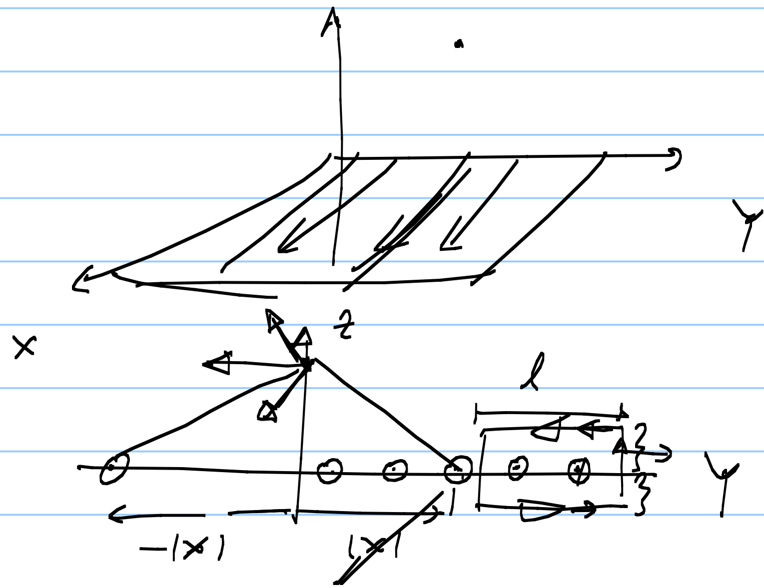
$$\boxed{\oint_C B \cdot d\vec{r} = \mu_0 I_{enc}}$$

Ampere's Law

$$\boxed{\begin{aligned} \nabla \cdot B &= 0 \\ \nabla \times B &= \mu_0 J \end{aligned}}$$

Application of Ampere's Law

<u>Ex</u> Sheet of current with surface current density (uniform) $\vec{K}$



(1) $\vec{B}(z)$ only and in $-\hat{y}$ direction

$z > 0$

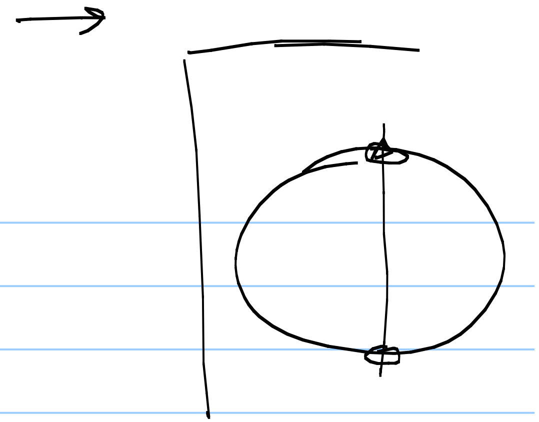
$+\hat{y}$ direction

$z < 0$

$$B(z)\cdot l + B(z)\cdot l = \mu_0 K \cdot l$$

$$\vec{B}(z) = \frac{\mu_0 K}{2}(-\hat{y}) \qquad z > 0$$

$$= \frac{\mu_0 K}{2}\hat{y} \qquad z < 0$$



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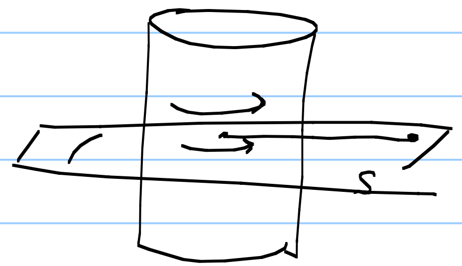
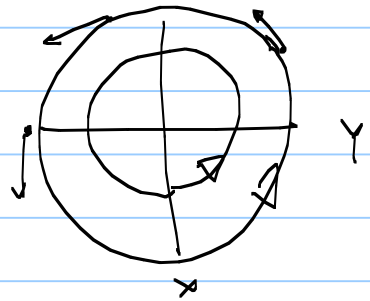
<u>Ex</u> Solenoid, n turns/length, current I



$\vec{B}(s)$

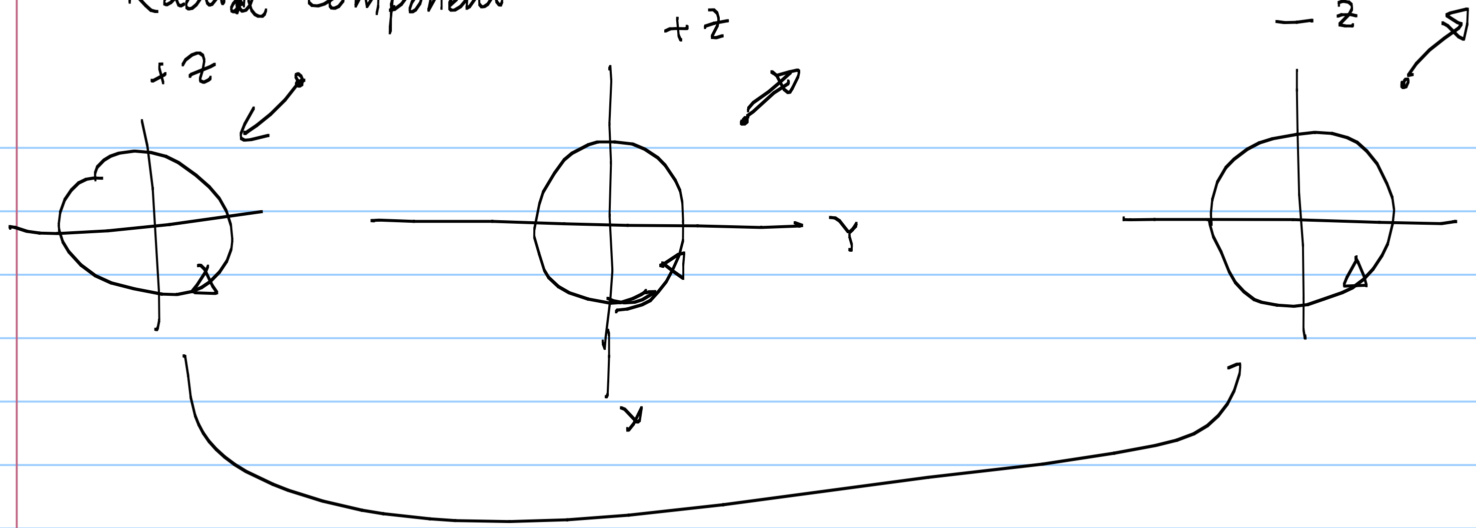
Can be have $\hat{\phi}$ component?



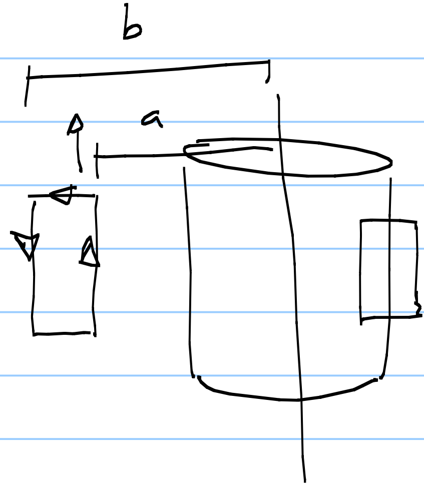
$\oint \bar{B}\cdot \bar{dl} > 0$ if $B_\phi > 0$

$\Rightarrow B_\phi = 0$

Radial component



$$\Rightarrow \quad B_r = 0$$



$$\vec{B} = B_z(s)\,\hat{z}$$

$$B(a)\cdot l + 0 + (-B(b))\cdot l + 0 = 0$$

$$B(a) = B(b) = B(\infty) = 0$$

Inside $\vec{B} = \mu_0 n I\,\hat{z}$ outside $\vec{B} = 0$

Magnetic Vector Potential

In ES: $\nabla \times \underline{E} = 0 \Rightarrow \bar{E} = -\nabla\phi$ $\qquad \nabla \times (\nabla \phi) = 0$

MS: $\nabla \cdot B = 0 \Rightarrow \bar{B} = \bar{\nabla} \times \bar{A}$ $\qquad \nabla \cdot (\bar{\nabla} \times \bar{F}) = 0$

Magnetic Vector potential.

– $\vec{A} \longrightarrow$ Theoretical Importance

In ES $\qquad$ Electric Potential not unique

$$\underline{E} \longrightarrow V_1(\vec{r})$$

$$V_2(\vec{r}) = V_1(\vec{r}) + C$$

In MS $\qquad B \longrightarrow A_1(\vec{r})$ $\qquad$ A is not unique

$$A_2(\vec{r}) = A_1(\vec{r}) + \nabla\lambda$$

λ : any scalar field

$$\nabla \times A_2 = \nabla \times A + \nabla \times (\nabla \lambda)$$
$$= B$$

choose divergence less vector potential, choose A s.t

$$\nabla \cdot A = 0$$

suppose $\quad \nabla \cdot A_1 \neq 0$

$$\nabla \cdot A_2 = 0 = \nabla \cdot A_1 + \nabla \cdot (\nabla \lambda)$$

$$\nabla^2 \lambda = - \underbrace{\nabla \cdot A_1} = -\varphi$$

Poisson Eq.

Differential Eq for Vector potential

$$\nabla \times B = \mu_0 J$$

$$\Rightarrow \quad \nabla \times (\nabla \times A) = \mu_0 J$$

$$\Rightarrow \quad \nabla (\nabla \cdot A) - \nabla^2 A = \mu_0 J$$

$$\Rightarrow \quad \nabla^2 \bar{A} = -\mu_0 \bar{J} \qquad \nabla \cdot A = 0$$

Three Poisson Eq.

$$\Rightarrow \quad \begin{aligned} \nabla^2 A_x &= -\mu_0 J_x \\ \nabla^2 A_y &= -\mu_0 J_y \\ \nabla^2 A_z &= -\mu_0 J_z \end{aligned}$$

Only in Cartesian Co-ordinates

$\cancel{\nabla^2 A_r = -\mu_0 J_r}$

$$A_x = \frac{\mu_0}{4\pi} \int \frac{J_x(r')\,dv'}{|r-r'|}$$

$$\boxed{\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(r')\,dv'}{|r-r'|}}$$

In Es

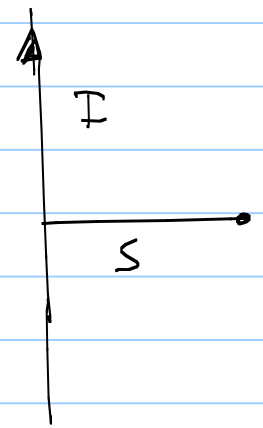
$$\nabla^2 V = -\frac{\rho}{\epsilon_0}$$

$$\downarrow$$

$$\boxed{V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(r')\,dv'}{|r-r'|}}$$

Ex

$$\vec{A} = A_z(s)\,\hat{z}$$



$$\vec{\nabla} \times A = -\hat{\phi}\,\frac{\partial A_z}{\partial s} = \frac{\mu_0 I}{2\pi s}\,\hat{\phi}$$

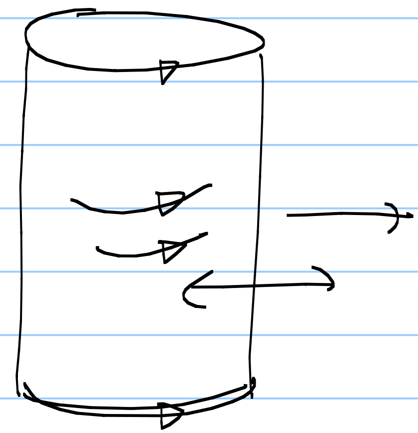
$$A_z = -\frac{\mu_0 I}{2\pi} \ln s$$

$$\vec{A} = -\frac{\mu_0 I}{2\pi} \ln s\;\hat{z}$$

$$\bar{\nabla}\cdot\bar{A} = \frac{\partial}{\partial z} A_z = 0$$

Ex

$$\vec{B} = \mu_0 n I \hat{z} \quad \text{inside}$$
$$= 0 \quad \text{outside}$$



$$\vec{A} = A_\phi(s)\,\hat{\phi}$$

$$\left(\bar{\nabla}\times\bar{A}\right)_\phi = \underbrace{\frac{1}{s}\frac{\partial}{\partial s}(sA_\phi)} = \mu_0 n I$$

$$A_\phi = \frac{\mu_0 n I s}{2}$$

$$\vec{A} = \frac{\mu_0 n I s}{2}\,\hat{\phi} \quad \text{inside}$$

$$\vec{A} = \frac{C}{s}\,\hat{\phi} \quad \text{outside}$$

$$= \frac{\mu_0 n I a^2}{2 s}\,\hat{\phi}$$

outsid

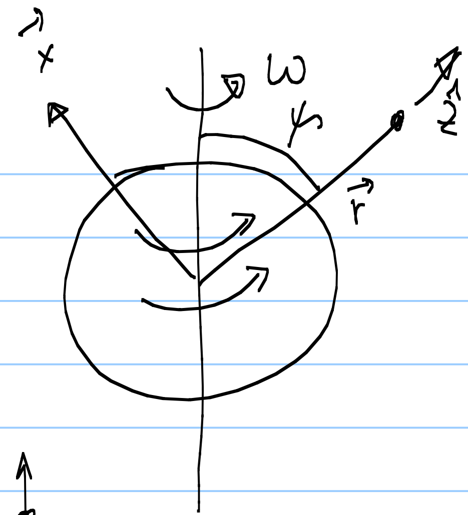
$$\frac{1}{s}\frac{\partial}{\partial s}(sA_\phi) = 0$$

$$sA_\phi = C$$

$$A_\phi = \frac{K}{s}$$

Ex Sphere with surface charge density σ

Sphere is spinning with ang. Vel. $\vec{\omega}$

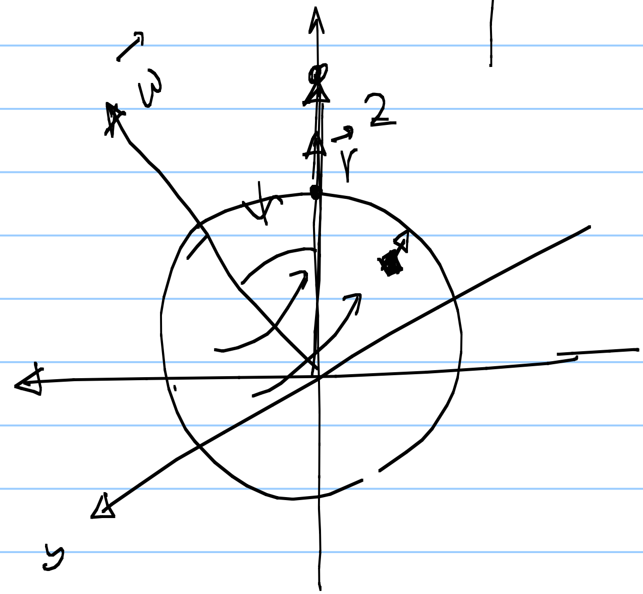


$\vec{r} = r\,\hat{z}$

$\vec{\omega} = (\omega \sin\psi\,, 0\,, \omega\cos\psi)$

$r' = (R, \theta', \phi')$ in sph

$= (R\sin\theta'\cos\phi', R\sin\theta'\sin\phi', R\cos\theta')$



$da' = R^2 \sin\theta\, d\theta\, d\phi$

$\vec{v}$ at $r' = \vec{\omega} \times \vec{r}'$

$= [\hat{x}\,(-\cos\psi \sin\theta' \sin\phi')$

$+ \hat{y}\,(-\sin\psi\cos\theta' + \sin\theta'\cos\phi'\cos\psi)$

$+ \hat{z}\,(\sin\psi \sin\theta' \sin\phi')\,]\,R\omega$

$\vec{K} = \sigma\vec{v} = \sigma R\omega[\qquad]$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{K(r')\, da'}{|r - r'|} \qquad d\phi'$$

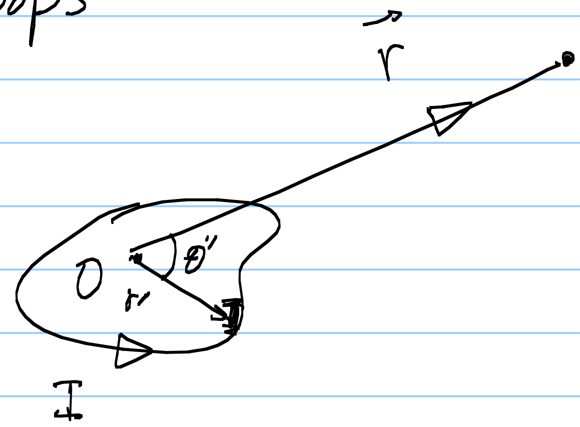
$$= \frac{-\mu_0}{4\pi} \sigma R^2 \omega \int \frac{\hat{\varphi}\,(\sin\psi \cos\theta')\, \sin\theta' d\theta' d\phi'}{(r^2 + R^2 - 2Rr\cos\theta')^{1/2}}$$

Vector Potential

Note Title 4/4/2009

Multipole Expansion $1/r$ only loops

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \oint \frac{I \quad \vec{dl}'}{|\bar{r}-\bar{r}'|} \qquad \vec{dl}' \rightarrow K da'$$



$$\frac{1}{|\bar{r}-\bar{r}'|} = \frac{1}{r} \sum_{n=0}^{\infty} \left(\frac{r'}{r}\right)^n P_n(\cos\theta')$$

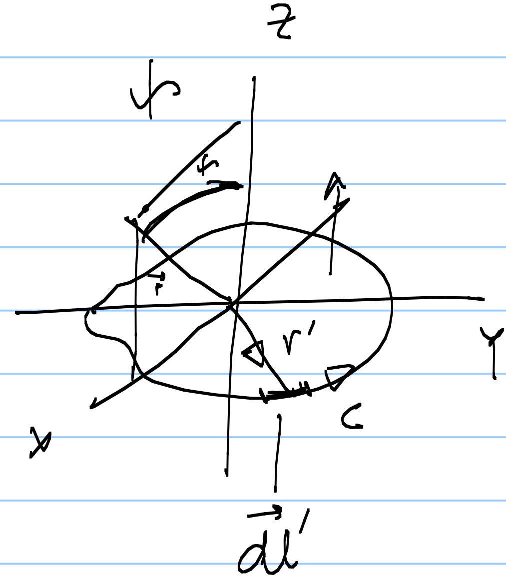
$$= \frac{1}{r} + \frac{r'}{r^2}\cos\theta' + \frac{r'^2}{r^3}\left(\frac{3\cos^2\theta' - 1}{2}\right) + \cdots$$

$$\vec{A}(\vec{r}) = \underbrace{\frac{\mu_0 I}{4\pi r} \oint \vec{dl}}_{\text{Monopole term}} + \underbrace{\frac{\mu_0 I}{4\pi r^2} \oint r'\cos\theta' \vec{dl}'}_{\text{dipole term}} + \underbrace{\cdots}_{\text{Q'pole term}}$$

Monopole term $= 0$ $\Rightarrow$ Monopole Moment $= 0$

Monopole don't Exist!

$$A_{dip}(\vec{r}) = \frac{\mu_0 I}{4\pi r^2} \oint r' \cos\theta' \, d\vec{l}' \Rightarrow$$



Planar loop: C in XY plane $(r', \theta = \frac{\pi}{2}, \phi')$

Angle between $\vec{r}$ and $\vec{r}'$

$$\cos\theta' = \sin\psi \cos\phi'$$

$$\vec{dl}' = dx' \hat{x} + dy' \hat{y}$$

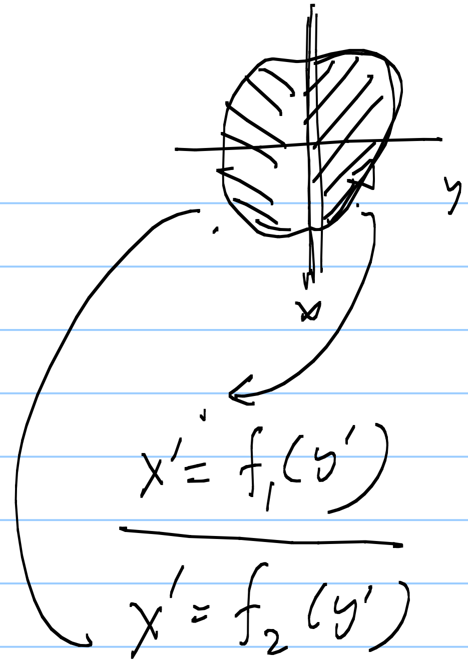
$$\vec{A}_{dip} = \frac{\mu_0 I \sin\psi}{4\pi r^2} \oint r' \cos\phi' (dx' \hat{x} + dy' \hat{y})$$

$$\vec{r} = (r\sin\psi, 0, r\cos\psi)$$

$$\vec{r}' = (r'\cos\phi', r'\sin\phi', 0)$$

$$= \frac{\mu_0 I \sin\psi}{4\pi r^2}\left[\hat{x}\oint x'\,dx' + \hat{y}\oint x'\,dy'\right]$$

$$= \frac{\mu_0 I \sin\psi}{4\pi r^2}\Big[0 + \hat{y}\underbrace{\oint x'\,dy'}_{\text{total area of loop} = a}\Big]$$



$$= \frac{\mu_0 \; I a \; \sin\psi \; \hat{y}}{4\pi r^2}$$

$\vec{m} = I(a\hat{z})$ = dipolemoment

$$\underbrace{= \frac{\mu_0}{4\pi}\,\frac{\vec{m}\times\hat{r}}{r^2}}_{\text{Co-ordinate free form}}$$

$\vec{r} = \hat{x}\, r\sin\psi + \hat{z}\, r\cos\psi$

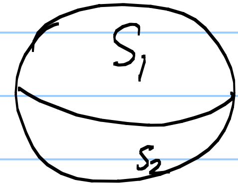
Non planar loops (tut problem)

$$A_{dip}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2}$$

$$\vec{m} = I\int_S d\vec{a}$$ = magnetic dipole mom

$S_1 + S_2$ is closed

$$\int_{S_1+S_2} d\vec{a} = 0$$



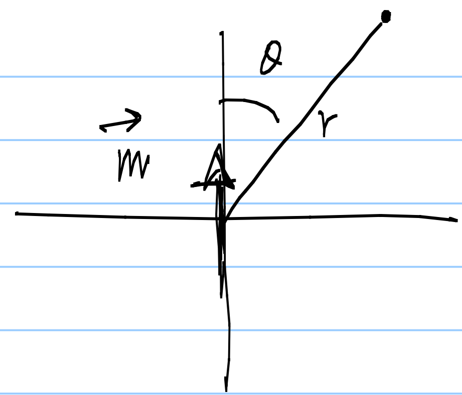
$$\oint da = 4\pi R^2$$

$$\int_{S_1} d\vec{a} = -\int_{S_2} d\vec{a}$$

$$\oint d\vec{a} = 0$$

Pure dipole is where only dipole term is nonzero

$$\vec{m} = m\hat{z}$$

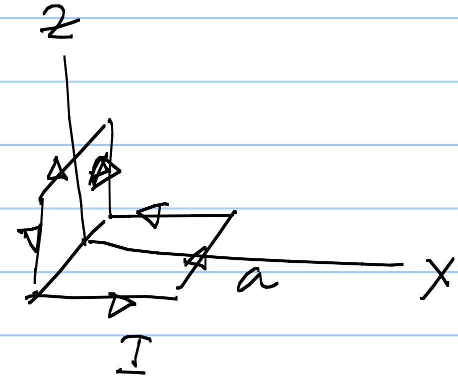


$$A(r,\theta,\phi) = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\hat{\phi}$$

$$B(r,\theta,\phi) = \nabla\times A$$

$$= \frac{\mu_0}{4\pi}\frac{m}{r^3}\left[\hat{r}\,2\cos\theta + \hat{\theta}\sin\theta\right]$$

$\vec{m}$ is additive

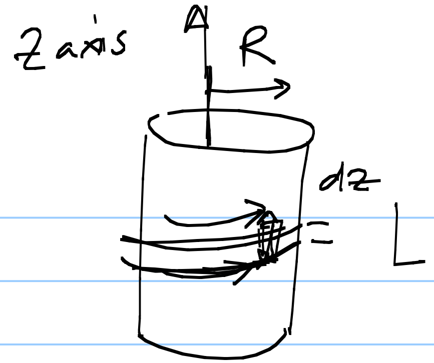


Ex $\vec{m} = Ia^2\left[\hat{z} + \hat{x}\right]$

<u>Ex</u> Solenoid of L and

nI

$\vec{m} = nI\pi R^2 L \hat{z}$



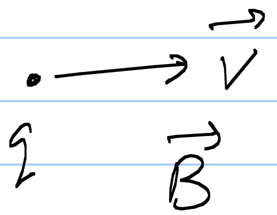
$$\left[\int da' \int d\bar{l}' \right] dz$$

Lorentz Force Law

Expt Fact (Law)

A point charge in magnetic field

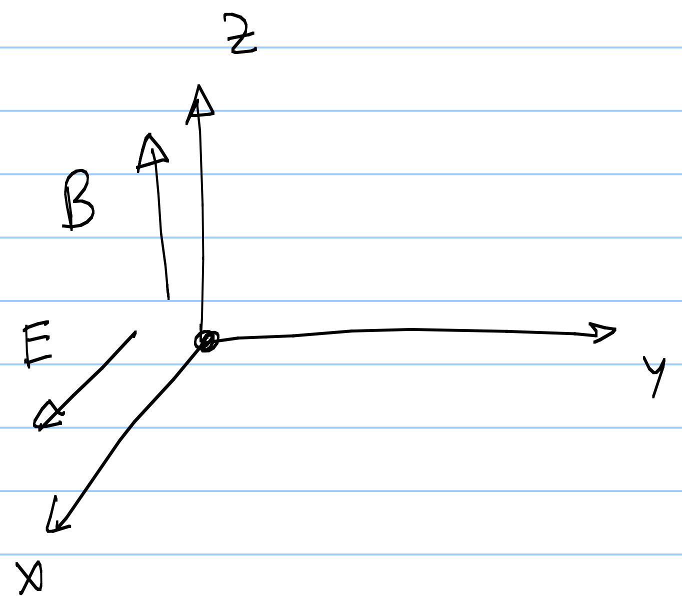
$F_{mag} = q(\vec{v} \times \vec{B})$



if $\underline{E}$ present

$$F_L = q\left(E + \bar{v} \times \vec{B}\right)$$

<u>Ex</u>



Given $\vec{B} = B\hat{z}$ $E = E\hat{x}$

Point charge q, mass m is released from origin.

Find the trajectory:

<u>point charges</u>

$$\cancel{\vec{B} = \frac{\mu_0}{4\pi} \frac{q v \times (r - r')}{|r - r'|}}$$

if a point charge q has velocity $\vec{v}$

Field due to pt. charge

?

$$\vec{r} = (x(t), y(t), z(t)) \qquad \vec{v} = (v_x(t), v_y(t), v_z(t))$$

$$m\vec{a} = q(E + \vec{v} \times \vec{B})$$

$$\vec{v} \times \vec{B} = \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ v_x & v_y & v_z \\ 0 & 0 & B \end{bmatrix}$$

$$= \hat{x}(v_y B) - v_x B \hat{y}$$

$$m(\ddot{x}\hat{x} + \ddot{y}\hat{y} + \ddot{z}\hat{z}) = \hat{x}(qE + qv_y B) + (-v_x B q)\hat{y}$$

$$\ddot{z} = 0 \Rightarrow z = At + B \qquad z(0) = 0, \quad \dot{z}(0) = 0$$

$$z = 0 \quad \forall t$$

$$\dot{v}_y = -\frac{qB}{m} v_x$$

$$\dot{V}_x = \frac{qE}{m} + \frac{qB}{m} V_y$$

$$\ddot{V}_x = \left(\frac{qB}{m}\right) \dot{V}_y = -\left(\frac{qB}{m}\right)^2 V_x = -\omega^2 V_x$$

$$V_x = A\sin(\omega t) + B\cos(\omega t)$$ $V_x(0) = 0$

$$= A\sin(\omega t)$$

$$V_y = \frac{A\omega\cos\omega t}{\omega} - \frac{qE}{m\omega}$$

$$= A\cos\omega t - \frac{E}{B}$$ $V_y(0) = 0$

$$= \frac{E}{B}(\cos\omega t - 1)$$

$$\dot{x} = \frac{E}{B}\sin\omega t$$ $x(t) \rightarrow$ Cycloid.