# Gauss Law for Dielectric Materials

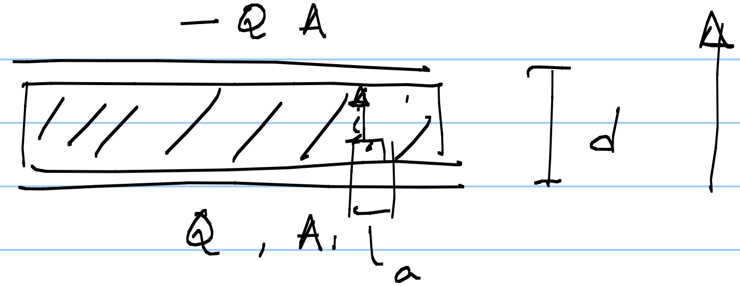
Note Title 3/25/2009

$$\underline{\nabla \cdot \bar{D} = \rho_f} = \text{free charge}$$

with $\bar{D}(\vec{r}) = \epsilon_0 E(\vec{r}) + P(\vec{r})$  E : total field at $\vec{r}$

Integral Form $\oint_S \bar{D} \cdot \hat{n}\, ds = Q_{f,enc}$ = Free charge enclosed in S

Ex  Translational symmetry in X, Y




D, E, P depend Z co-ordinate $\hat{z}$

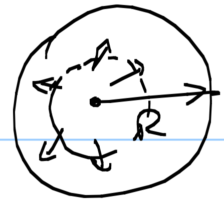
$$D \cdot a = (Q/A)\, a \Rightarrow \vec{D} = \frac{Q}{A}\hat{z} \quad \text{if} \quad \vec{P} = P_0 \hat{z} \text{ given}$$

$$\epsilon_0 \vec{E} = \vec{D} - \vec{P}$$

Ex "Frozen in" $\vec{P} = \frac{k}{r}\hat{r}$ $r < R$

$= 0$ $r > R$

No free charges




P is spherically symmetric

$\sigma_b$ : must be sph. symmetric

D, E also sph. symmetric $\Rightarrow$ $\hat{r}$ direction and depend on r only

$$4\pi r^2 |\vec{D}| = 0$$

$$\vec{D} = 0 \quad \text{everywhere}$$

$$\vec{E} = \frac{1}{\epsilon_0}(\bar{D} - \bar{P}) = -\frac{\vec{P}}{\epsilon_0} = -\frac{k}{\epsilon_0 r}\hat{r}$$

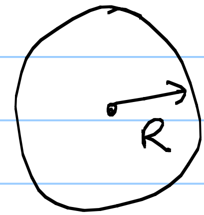
Alternate

$$\underbrace{\sigma_b, \ \rho_b}$$
$$\Downarrow$$
$$\vec{E}$$

Ex (a) A point charge in vacuum

$$\vec{E} = \frac{q\,\hat{r}}{4\pi\epsilon_0 r^2} \qquad \vec{D} = \frac{q\,\hat{r}}{4\pi r^2}$$

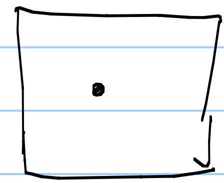
(b) A point charge surrounded sph dielectric




$$\vec{D} = \frac{q\,\hat{r}}{4\pi r^2}$$

Sph symmetry

(c) A point charge, surrounded by cubic dielectric



$$\vec{D} = ?$$

$$\Rightarrow \quad \nabla \cdot (\epsilon_0 \vec{E}) = \rho \qquad \vec{E} = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(r')\,dv'\,(\vec{r}-\vec{r}')}{|\vec{r}-r'|^3}$$

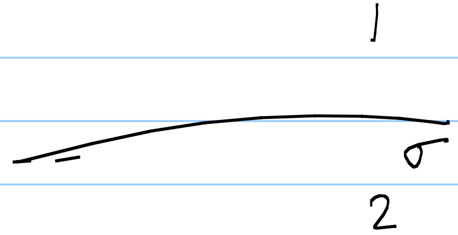
Q? $\quad \nabla \cdot D = \rho_f \qquad \vec{D} \overset{?}{=} \frac{1}{4\pi} \int \frac{\rho(r')\,dv'\,(r-r')}{|r-r'|^3}$ ?

$\times$

$$\vec{\nabla} \times \vec{E} = 0$$

$$\nabla \times D = \nabla \times (\epsilon_0 E + P)$$
$$= + \nabla \times P \neq 0$$

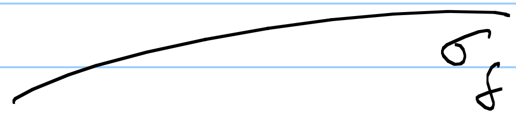
BC for E Accross a surface




$$E_{\perp,1} - E_{\perp,2} = \sigma / \epsilon_0$$

$$E_{\parallel,1} = E_{\parallel,2}$$

BC for D




$$D_{\perp,1} - D_{\perp,2} = \sigma_f$$

$$D_{\parallel,1} - D_{\parallel,2} = P_{\parallel,1} - P_{\parallel,2}$$

# Linear Dielectrics

$$\left.\begin{array}{l}\nabla\cdot D = \rho_f \\ \nabla\times E = 0\end{array}\right\}$$

In vac

$$\left.\begin{array}{l}\nabla\cdot E = \dfrac{\rho}{\epsilon_0} \\ \nabla\times E = 0\end{array}\right\}$$

def: $D = \epsilon_0 E + P$

$\longrightarrow P(E) \longleftarrow$ Constitutive relation

need to know this

Expt Fact: For large class of materials if $E$ is small

$$\vec{P} = \chi_e \epsilon_0 \vec{E} \qquad \vec{P}(\vec{E}) = 0 + \alpha E + \beta E^2 +$$

$\chi_e$ : Electric Susceptibility

Linear ( linear relation $\vec{P}$ and $\vec{E}$ ), homogeneous material ($\chi_e$ is ind of position), isotropic ( ind of direction)

$$\vec{P} = \begin{bmatrix} 0 & & \\ & 0 & \\ & & 0 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$ non isotropic

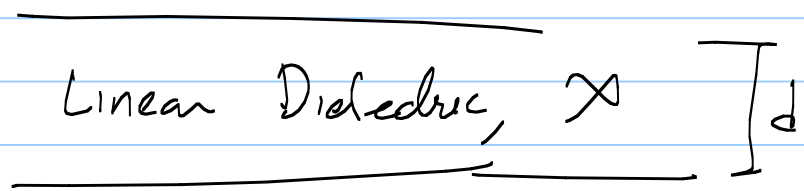
$$\vec{D} = \epsilon_0 \bar{E} + \vec{P} = \epsilon_0 (1+\chi_e) \vec{E} = \epsilon \vec{E}$$

$\epsilon$: Permittivity of material.

$$K = \frac{\epsilon}{\epsilon_0} = 1+\chi_e = \text{Dielectric constant}$$

Ex. Parallel plate capacitor (all $\hat{z}$ direction)

$$D = \frac{Q}{A}$$

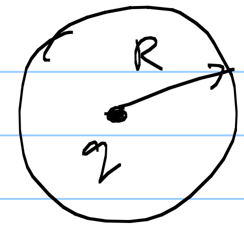



$$E = \frac{Q}{A\epsilon} \quad , \quad P = \frac{\chi_e Q}{(1+\chi_e) A}$$

Capacitance $C = \dfrac{A\epsilon}{d}$

Ex

Point charge in spherical, linear Dielectric, susc $\chi_e$.




$$\vec{D} = \frac{q}{4\pi r^2}\hat{r} \quad \text{everywhere}$$

$$\vec{E} = \frac{1}{\epsilon}\vec{D} = \frac{q\hat{r}}{4\pi\epsilon r^2} \quad r<R$$

$$= \frac{1}{\epsilon_0}\vec{D} = \frac{q\hat{r}}{4\pi\epsilon_0 r^2} \quad r>R$$

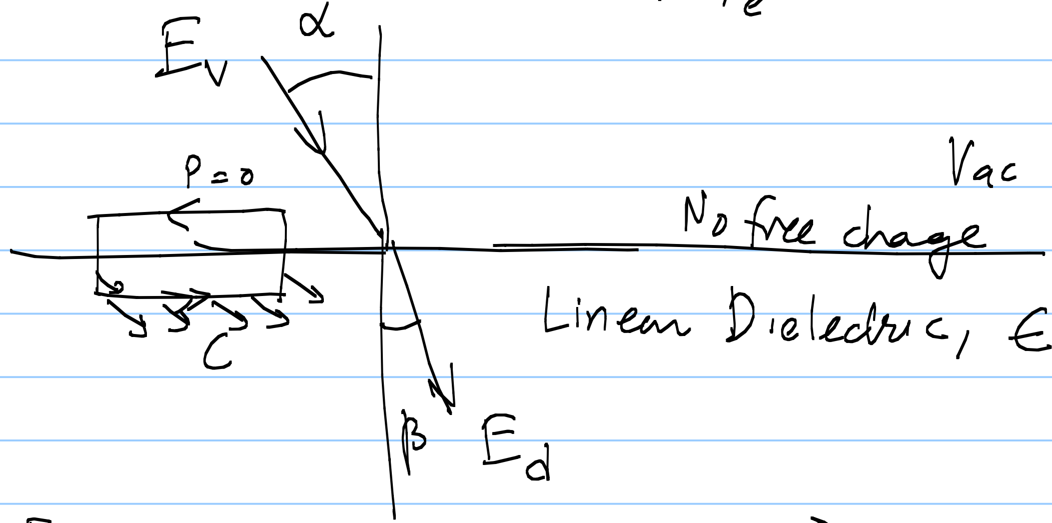
$$\vec{P} = \chi_e\epsilon_0\vec{E} = \frac{\chi_e q\hat{r}}{(1+\chi_e)4\pi r^2} \quad r<R$$

$$= 0 \quad r>R$$

$$\sigma_b = \frac{\chi_e q}{(1+\chi_e)4\pi R^2} \qquad Q_{b,\text{surface}} = \frac{\chi_e q}{1+\chi_e}$$

$$\rho_b = -\vec{\nabla}\cdot\vec{P} = -\frac{\chi_e q}{(1+\chi_e)4\pi}\left(\nabla\cdot\frac{\hat{r}}{r^2}\right) = -\frac{\chi_e q}{(1+\chi_e)4\pi}\left(4\pi\,\delta^3(\vec{r})\right)$$

$$q_b = -\frac{\chi_e q}{1+\chi_e}$$




Ex

Use BC

$$D_{\perp,v} = D_{\perp,d}$$

$$E_{\parallel,v} = E_{\parallel,d}$$

$$\epsilon_0 E_v \cos\alpha = \epsilon E_d \cos\beta$$

$$E_v \sin\alpha = E_d \sin\beta$$

$$\frac{1}{\epsilon_0} \tan\alpha = \frac{1}{\epsilon} \tan\beta$$

$\vec{P}$

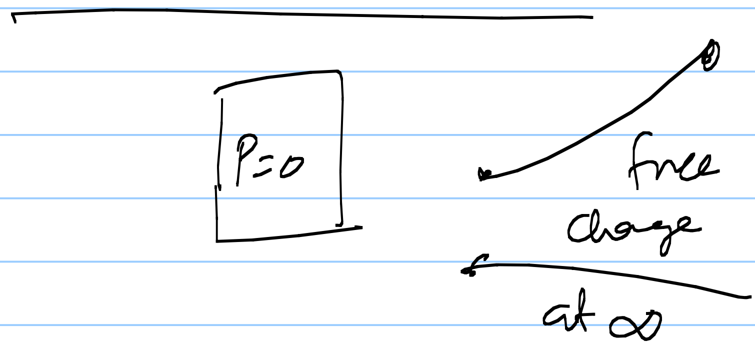
$$\oint_C P \cdot dl > 0$$

$\Rightarrow \nabla \times P \neq 0$ everywhere

# Energy in Dielectrics

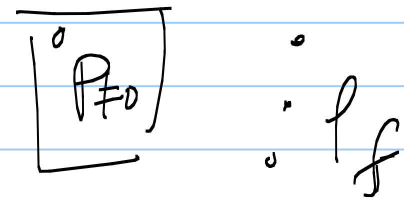
Energy is work done in bringing free charge to their locations.

$$\Delta\rho_f \longrightarrow \Delta D$$




$$\Delta W = \int (\Delta\rho_f)\, V\, d\tau$$

$$= \int_V \nabla\cdot(\Delta D)\, V\, d\tau$$




$$= \int_V \left[\nabla\cdot(\Delta D V) - \Delta D\cdot\nabla V\right] d\tau$$

$$= \oint_S V\, \Delta\vec{D}\cdot\hat{n}\, ds + \int_V \Delta D\cdot E\, d\tau$$

$$\nabla\cdot(f\bar{A}) = (\nabla\cdot\bar{A})f + \nabla f\cdot\bar{A}$$

$$\Downarrow$$

$$0$$

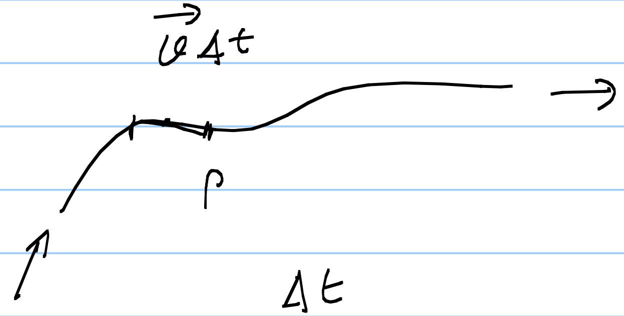
$$= \int_{\substack{\text{Entire} \\ \text{Space}}} \Delta \underline{D} \cdot \underline{E} \; d\tau$$

If Dielectric is linear $\quad \Delta(\underline{D} \cdot \underline{E}) = \Delta \underline{D} \cdot \underline{E} + \underline{D} \cdot \Delta \underline{E}$

Note Title 3/26/2009

# Magnetostatics

## Currents:

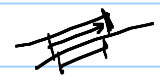



$I = \frac{dq}{dt}$ = Amount of charge that crosses p in unit time

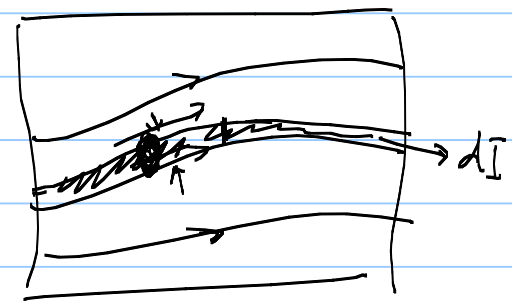
$$\vec{I} = \frac{\lambda \vec{v} \Delta t}{\Delta t} \quad \text{in time } \Delta t$$

$$= \lambda \vec{v}$$

$\lambda$ : charge density of carriers



width of one strip $dl_\perp$

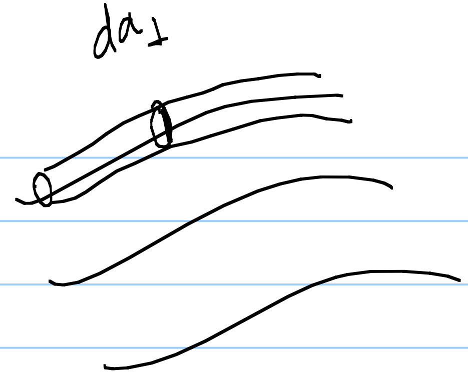



$$\vec{K} = \frac{d\vec{I}}{dl_\perp} = \text{surface current density}$$

$$= \sigma \vec{v}$$

$\sigma$ : charge density of carriers

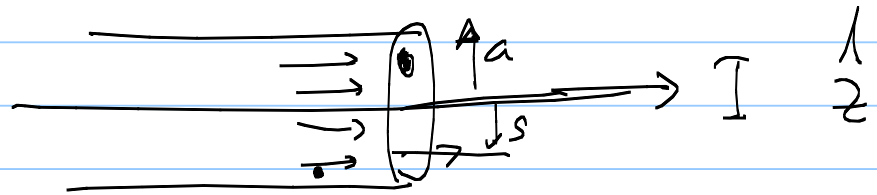
Volume current density:




$$\vec{J} = \frac{d\vec{I}}{da_\perp}$$

$$= \rho \vec{v}$$

Ex Thick wire, radius $a$, net $I$




(a) $\vec{K} = \frac{I}{2\pi a} \hat{z}$ if current is uniformly distributed over surface

(b) $\vec{J} = \frac{A}{s} \hat{z}$

$$\text{net current} = \int \vec{J}\cdot\hat{n}\,da \qquad \hat{n} = \hat{z}$$

$$da = (s\,d\phi)\,ds$$

$$= \int_0^a\int_0^{2\pi} \frac{A}{s}\, s\,d\phi\,ds$$

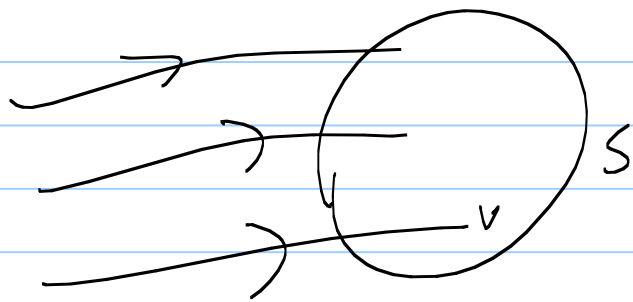
$$= A\cdot 2\pi a = I \quad\Rightarrow\quad A = \frac{I}{2\pi a}$$

$$\vec{J} = \frac{I\,\hat{z}}{2\pi a s}$$

Continuity Eq.

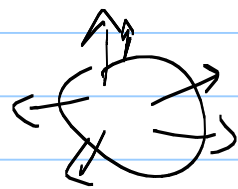
Conservation of charges

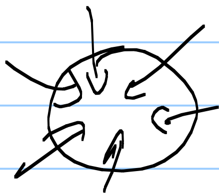
$$\oint_S \vec{J}\cdot\hat{n}\,ds = -\frac{dQ}{dt}$$




$$= -\frac{d}{dt}\underbrace{\int_V \rho(\vec{r},t)\,dv}$$

$$= -\int_V \frac{\partial\rho}{\partial t}\,dv$$





$$\Rightarrow \quad \int_V (\bar{\nabla}\cdot\bar{J})\, dv = -\int_V \left(\frac{\partial \rho}{\partial t}\right) dv$$

$$\left(\vec{\nabla}\cdot\vec{J} + \frac{\partial \rho}{\partial t}\right) = 0$$

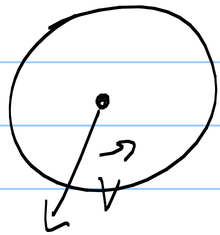
Continuity eq.
Cons. of charges

Steady current: $\rho$ is independent of $t$

$$\Rightarrow \quad \nabla\cdot J = 0$$

Magnetostatics
→ steady currents
→ $\frac{v}{c} \ll 1$

Ex



$q$ at $r_0$

$$\rho(\vec{r}) = q\,\delta(\vec{r}-\vec{r}_0) \qquad \bar{r}_0 = \vec{v}\,t$$

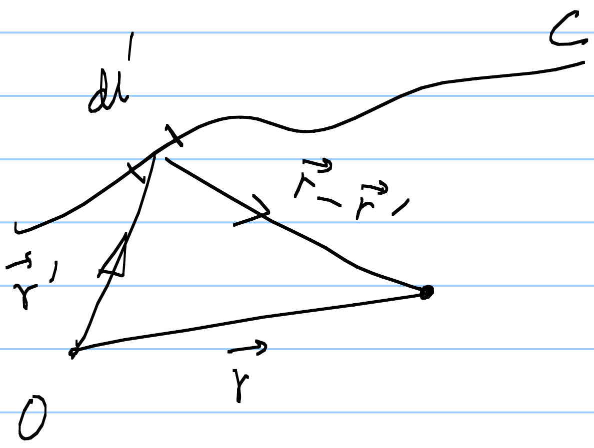
$$\rho(\vec{r},t) = q\,\delta(\vec{r}-v_0 t)$$

# Magnetic of steady currents

**Biot-Savart Law:** A wire carrying steady current $I$

Magnetic field at $\vec{r}$

$$B(\vec{r}) = \frac{\mu_0}{4\pi} \int_C \frac{I(\vec{r}\,') \times (\bar{r} - \bar{r}\,')}{|\bar{r} - \bar{r}\,'|^3} \, dl'$$




$$\mu_0 = 4\pi \times 10^{-7} \frac{N}{A^2} = \text{permeability of space} = \text{Exact number}$$

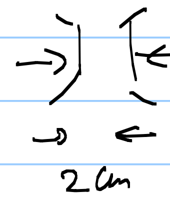
SI Unit for B

$$1\,T = 1 \frac{N}{A \cdot m}$$

$$= 10^4 G$$

Lab

$\sim 3000$ G

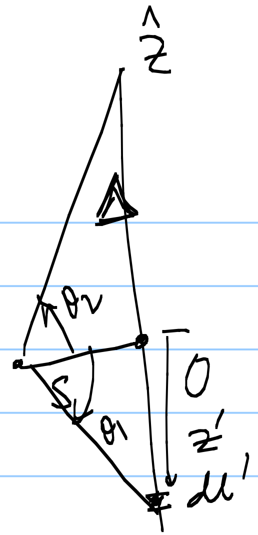


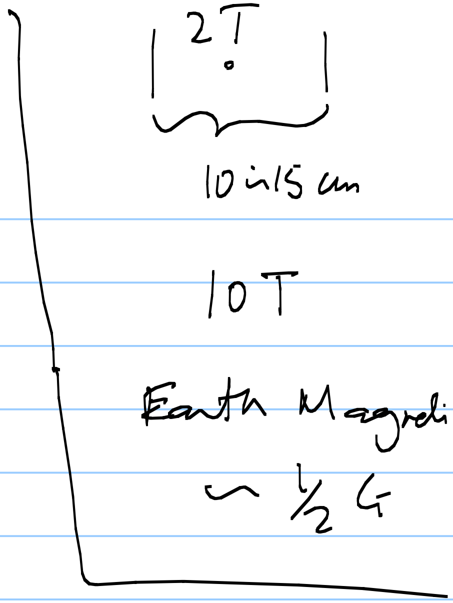
**Ex** straight wire carrying current $I$

$$\vec{r} = s\,\hat{s} \qquad \vec{I} = I\,\hat{z}$$

$$\vec{r}\,' = z\,\hat{z}$$

$$dl' = dz'$$




$$\vec{I} \times (\vec{r} - \vec{r}\,') = I\,\hat{z} \times \left(\underline{z'\hat{z} - s\,\hat{s}}\right)$$

$$= 0 - Is\left(-\hat{\phi}\right) = Is\,\hat{\phi}$$

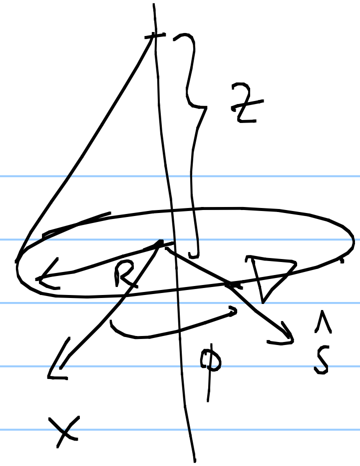
$$|\vec{r} - \vec{r}\,'| = \left(z'^2 + s^2\right)^{1/2}$$

$$B(s\,\hat{s}) = \frac{\mu_0}{4\pi} \int_{z_1'}^{z_2'} \frac{Is\,\hat{\phi}\,dz'}{\left(z'^2 + s^2\right)^{3/2}}$$

$$= \frac{\mu_0 I s}{4\pi} \;\; - - - - \qquad z = s\tan\theta$$

$$= \frac{\mu_0 I}{2\pi s}\left(\sin\theta_2 - \sin\theta_1\right)\hat{\phi} \quad \longleftarrow \text{Check.}$$

Ex Ring carrying current $I$ (HW)

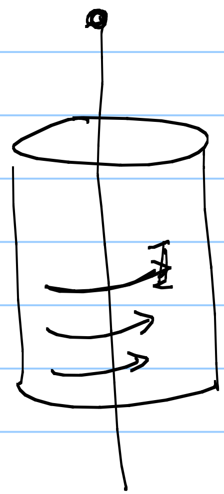



$$B = \frac{\mu_0 I}{2} \frac{R^2}{(R^2+z^2)^{3/2}}$$

$$\int_{\text{on wire}} \hat{s}\, d\phi = \int_0^{2\pi} \hat{s}(\phi)\, d\phi = 0$$

$$\hat{s} = \cos\alpha\, \hat{x} + \sin\alpha\, \hat{y}$$

Ex (HW) Solenoid, $n$ turns/length




$I$

$dl$

$\vec{K}$ = Surface current density

$$= \left( n\, dl\, I / dl \right) \hat{\phi}$$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{k}(\vec{r}\,') \times (\bar{r} - \bar{r}\,')}{|\bar{r} - \bar{r}\,'|^3} \, ds'$$

$$= \frac{\mu_0}{4\pi} \int \frac{J(r') \times (\bar{r} - \bar{r}\,')}{|r - r'|^3} \, dv'$$