Gauss Law for Dielectric Materials

Note Title 3/25/2009

$$\nabla \cdot D = \rho_f = \text{Free charges}$$

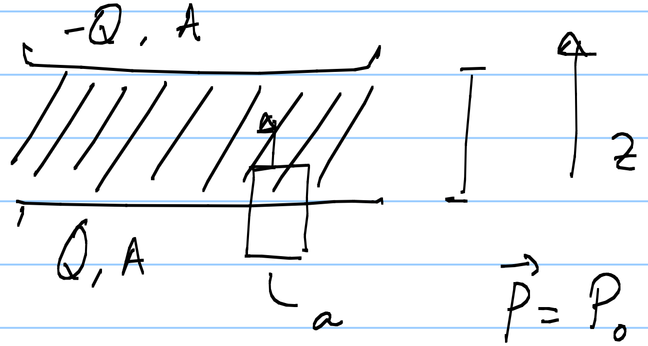
$$D(\vec{r}) = \epsilon_0 E(\vec{r}) + P(\vec{r}) \qquad E: \text{total Electric field}$$

Integral

$$\oint_S D\cdot\hat{n}\,ds = Q_{f,encl} = \text{Net free charge enclosed in } S.$$

Ex Translational symmetry

D, E, P fn of z only in $\hat{z}$ direction



$$\vec{P} = P_0\,\hat{z}$$

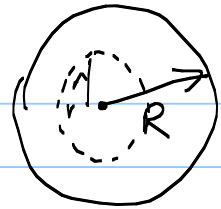
$$D\cdot a = \left(\frac{Q}{A}\right) a$$

$$\vec{D} = \frac{Q}{A}\hat{z}, \qquad \vec{E} = \frac{1}{\epsilon_0}\left(\frac{Q}{A}\hat{z} - P_0\hat{z}\right)$$

Ex "Frozen in" Polarization, No free charges

$$\vec{P} = \frac{k}{r}\hat{r} \qquad r < R$$

$$= 0 \qquad r > R$$



Spherical symmetry : $\vec{P}$, ρ_b, σ_b,

$\vec{D}$, $\vec{E}$ also must be spherically symmetry

$$D \cdot 4\pi r^2 = 0 \Rightarrow \vec{D} = 0 \quad \text{everywhere}$$

$$\vec{E} = -\frac{1}{\epsilon_0}\vec{P} = -\frac{k}{\epsilon_0 r}\hat{r} \qquad r < R$$

$$= 0 \qquad r > R$$

Ex (a) In Vac

$$D = \frac{q}{4\pi}\frac{\hat{r}}{r^2} \qquad E = \frac{q}{4\pi\epsilon_0}\frac{\hat{r}}{r^2}$$

(b) A point charge surrounded by dielectric

$$D \cdot 4\pi r^2 = q \Rightarrow \vec{D} = \frac{q\hat{r}}{4\pi r^2}$$

(c) A point charge surrounded by a cubic dielectric

$D = ?$ (calculation difficult)

$|D|$ on S may not be const

$$\oint_S D \cdot \hat{n}\, ds = q$$

$$\nabla \cdot (\epsilon_0 \bar{E}) = \rho \Rightarrow \vec{E} = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(r')(r-r')\,dv'}{|r-r'|^3}$$

$$\nabla \cdot D = \rho_f \not\Rightarrow \vec{D} = \cancel{\frac{1}{4\pi}} \int \frac{\rho_f(r')(r-r')\,dv'}{|r-r'|^3}$$

Eq.

$$\left.\begin{aligned} \nabla \cdot E &= \rho/\epsilon_0 \\ \nabla \times E &= 0 \end{aligned}\right\}$$

Helmholtz

$$\nabla \cdot D = \rho_f$$

$$\nabla \times D = \epsilon_0 (\underbrace{\nabla \times E}_{0}) + \nabla \times P$$

$$= \nabla \times P \neq 0$$

BC

①

$$E_{1,\perp} - E_{2,\perp} = \frac{\sigma}{\epsilon_0}$$

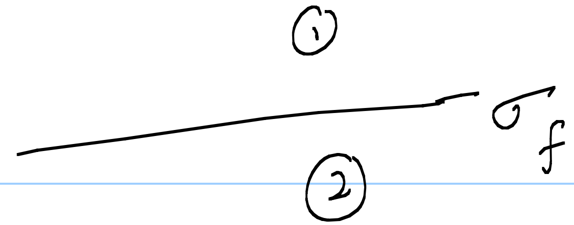
σ

②

$$E_{1,\parallel} = E_{2,\parallel}$$

$$D_{1,\perp} - D_{2\perp} = \sigma_f$$

$$D_{1,\parallel} - D_{2,\parallel} = P_{1,\parallel} - P_{2,\parallel}$$



Linear Dielectric

$$\nabla\cdot D = \rho_f \qquad D = \epsilon_0 E + P$$

$$\epsilon_0 \nabla\cdot E = \rho_f + \rho_b = \rho$$

$$\nabla\times E = 0$$

$P(E)$: Constitutive relation

Expt Fact: Large no of materials, if $|E|$ is small the:

$$\underline{\vec{P} = \chi_e \epsilon_0 \vec{E}}$$

χ_e: Electric Susceptibility

$$P(E) = 0 + \alpha E + \beta E^2 + \dots$$

Linear, homogeneous, isotropic

$$P(\vec{r}) = \chi_e(\vec{r})\,\epsilon_0 \bar{E}(\vec{r})$$

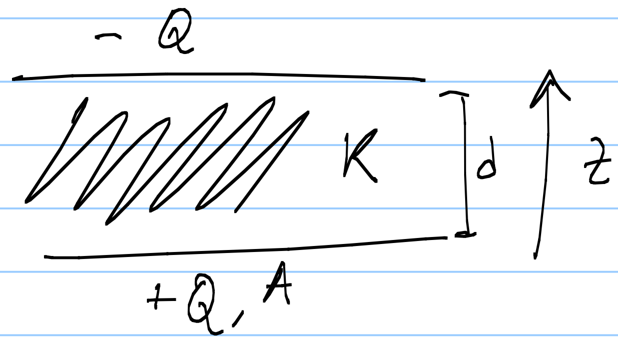
$$\vec{D} = \epsilon_0 E + P = \epsilon_0(\chi_e + 1)\bar{E}$$
$$= \epsilon \bar{E}$$

ϵ: permittivity of material

$$K = \frac{\epsilon}{\epsilon_0} = 1 + \chi_e = \text{Dielectric constant}$$

Ex

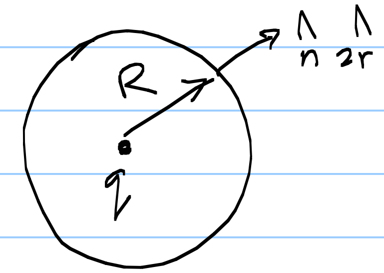
$$D = \frac{Q}{A} \qquad E = \frac{Q}{A\epsilon} \qquad E = \frac{1}{\epsilon} D$$



$$\vec{p} = \frac{\chi_e Q}{(1+\chi_e)A} = \frac{(K-1)Q}{KA} = \epsilon_0 \chi_e E$$

$$\text{Capacitance} = \frac{A\epsilon}{d}$$

Ex A point charge embedded at the center of a spherical linear dielectric



Symmetry : spherical.

$$\vec{D} = \frac{q}{4\pi}\frac{\hat{r}}{r^2} \qquad \text{Everywhere}$$

$$\vec{E} = \frac{1}{\epsilon}\vec{D} = \frac{q}{4\pi\epsilon}\frac{\hat{r}}{r^2} \qquad r<R$$

$$= \frac{1}{\epsilon_0}\vec{D} = \frac{q}{4\pi\epsilon_0}\frac{\hat{r}}{r^2} \qquad r>R$$

$$\vec{P} = \epsilon_0\chi_e\vec{E} = \frac{\chi_e q\,\hat{r}}{(1+\chi_e)4\pi r^2} \qquad r<R$$

$$= 0 \qquad r>R$$

$$\sigma_b = \vec{P}\cdot\hat{r}\Big|_{r=R} = \frac{\chi_e q}{(1+\chi_e)\,4\pi R^2}$$

Q_b on surface $= \dfrac{\chi_e q}{1+\chi_e}$

$$\rho_b = -\vec{\nabla}\cdot\vec{P} = -\frac{\chi_e q}{(1+\chi_e)4\pi}\,\nabla\cdot\left(\frac{\hat{r}}{r^2}\right)$$

$$= -\frac{\chi_e q}{1+\chi_e}\,\delta^3(\vec{r})$$

$\rho_f = \alpha\,\rho_b$

always

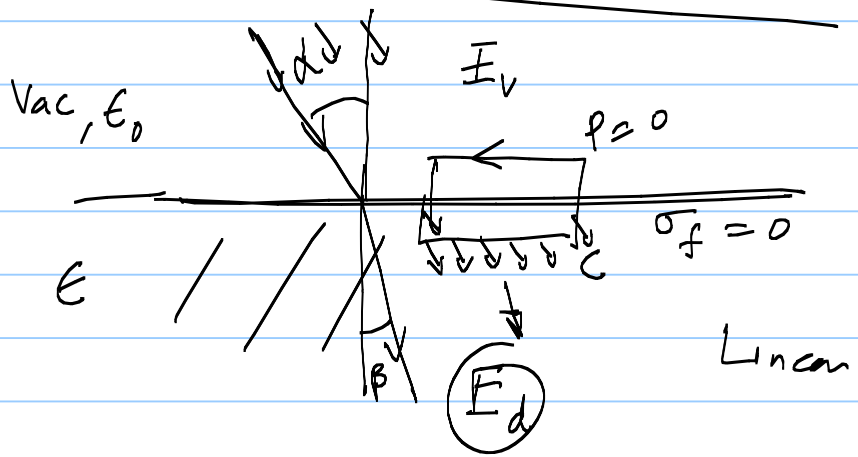
Ex BC

$$D_{v,\perp} = D_{d,\perp}$$

$$\epsilon_0 E_v \cos\alpha = \epsilon E_d \cos\beta$$

$$E_{v,\parallel} = E_{d,\parallel}$$

$$E_v \sin\alpha = E_d \sin\beta$$



$$\Rightarrow \quad \frac{1}{\epsilon_0}\tan\alpha = \frac{1}{\epsilon}\tan\beta$$

$$\oint_C \vec{P}\cdot d\vec{l} > 0$$

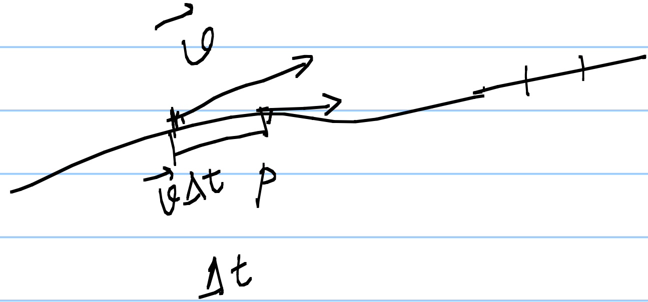
$$\nabla \times P \neq 0$$

$$\nabla \times D = \nabla \times (\epsilon E) \neq 0 \quad \text{everywhere}$$

Magnetism (Evening)

Note Title 3/26/2009

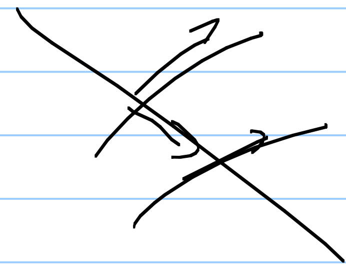
Current Densities:



Current at P, $I = \dfrac{dq}{dt}$ = charge crossing P in unit time

λ : charge density of carriers

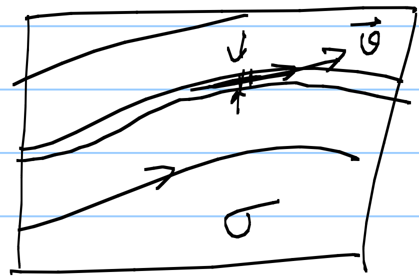
$$\vec{I} = \frac{\lambda \bar{v} \Delta t}{\Delta t} = \lambda \vec{v}$$



Surface current density

width of strip $= dl_\perp$

$\vec{K} = d\vec{I} / dl_\perp$



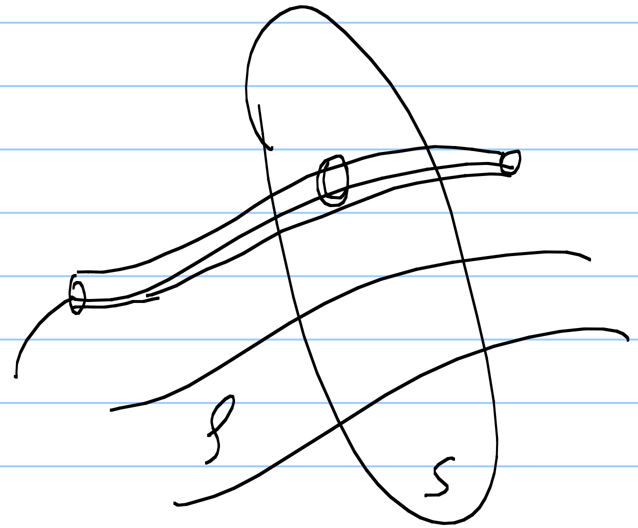
σ: surface charge density of carriers

$$\vec{K} = \frac{\sigma\, \bar{v}\Delta t\, dl_\perp}{\Delta t\, dl_\perp} = \sigma\vec{v}$$

Volume charge density

$$\vec{J} = \frac{d\bar{I}}{da_\perp}$$

$da_\perp$

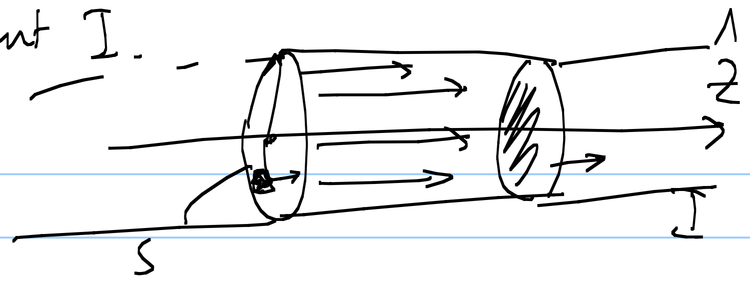


$$= \rho\vec{v}$$

Net current thro' S

$$I_{\substack{\text{net thro'}\\ S}} = \int J\cdot\hat{n}\, dS$$

Ex Thick wire, radius a, net current I.



(a) Current distributed over surface uniformly

$$\vec{K} = \frac{I}{2\pi a}\hat{z}$$

(b) Current is flowing thro' wire

$$\vec{J} = \frac{A}{s}\hat{z} \qquad \text{Find } A$$

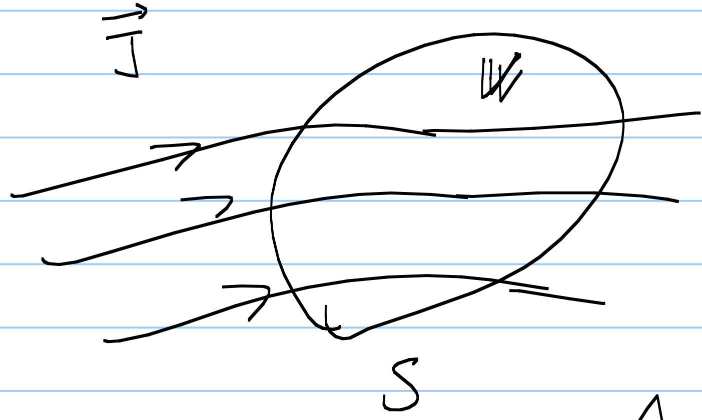
$$I = \int J\cdot\hat{n}\,ds \qquad ds = s\,ds\,d\phi$$

$$= \int_0^a\int_0^{2\pi}\frac{A}{s}\,s\,ds\,d\phi \qquad \hat{n} = \hat{z}$$

$$= A\cdot 2\pi a \qquad A = \frac{I}{2\pi a}$$

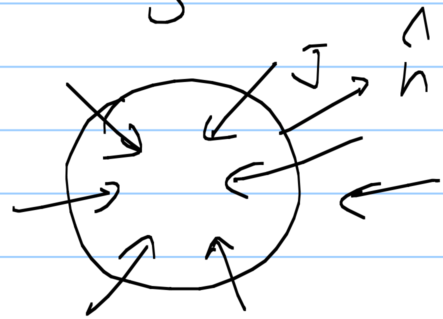
$$\vec{J} = \frac{I\,\hat{z}}{2\pi a s}$$

Continuity Eq :



$$\oint_S J \cdot \hat{n}\, ds$$

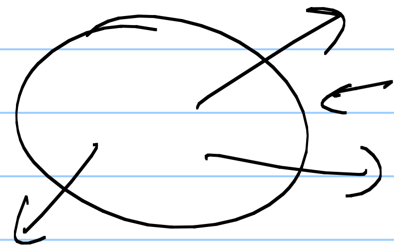
net charge flowing out of S in one unit time



Q : net charge in volume V

$$\oint_S J \cdot \hat{n}\, ds = -\frac{dQ}{dt}$$

if charges are conserved.



$$\int_V \vec{\nabla}\cdot\vec{J}\,dv = -\frac{d}{dt}\left(\int_V \rho(\vec{r},t)\,dv\right)$$

$$= \int_V \left(-\frac{\partial \rho}{\partial t}(\vec{r},t)\right) dv$$

$$\boxed{\nabla\cdot J = -\frac{\partial\rho}{\partial t}}$$ Continuity Eq

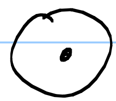
Steady Currents : ρ is ind of time

$$\nabla\cdot J = 0$$

point charge q, uniform velocity $\vec{v}$

$$\rho(\cancel{t}\,\vec{r}) = q\,\delta^3(\vec{r}-\vec{r}_0)$$

$$\rho(\vec{r},t) = q\,\delta^3(\vec{r}-\vec{v}\,t)$$

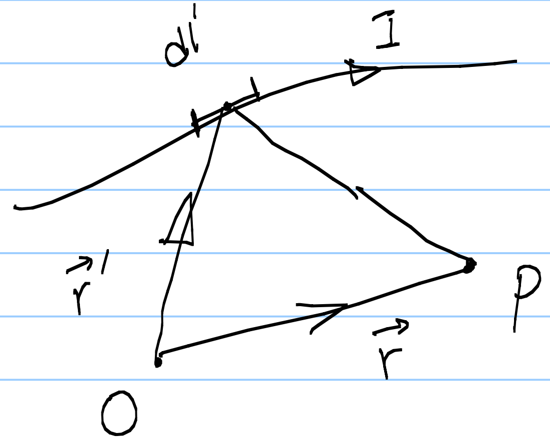


$t=0$ at $\vec{r}_0$

Magnetostatics

→ Steady currents

→ $\frac{v}{c} \ll 1$



Biot - Savart Law

A wire carrying current $\vec{I}$ (steady)

Magnetic field at $\vec{r}$

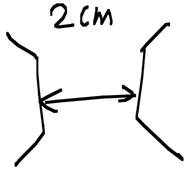
$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{I}(r') \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \, dl'$$

$$\mu_0 = 4\pi \times 10^{-7} \ N/A^2$$

SI unit $\quad 1\,T = 1 \frac{N}{A \cdot m} = 10^4 \, G$

$B_{Earth} \sim \frac{1}{2}\ G$

2 cm



0.5 T

2T

10 cm

10 T

Superconducting 15 T

Ex St wire, I,

$\vec{I} = I\hat{z}$

$\vec{r} = s\hat{s}$

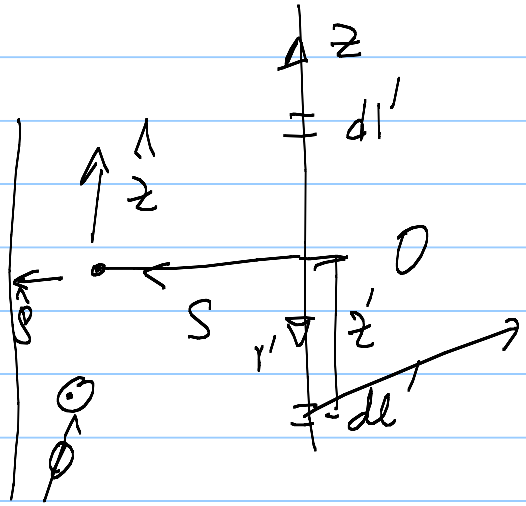
$r' = z'\hat{z}$

$dl' = dz'$

$I \times (\bar{r} - \bar{r}')$

$= I\hat{z} \times (s\hat{s} - z\hat{z})$

$= Is\hat{\phi}$

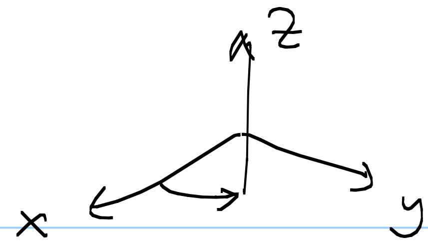


Co-ordinates $(0, 0, z')$

$$B(\vec{r}) = \frac{\mu_0}{4\pi} \int_{-\infty}^{\infty} \frac{Is\hat{\phi}\, dz'}{(s^2 + z'^2)^{3/2}}$$

$$|\bar{r} - \bar{r}'| = (s^2 + z'^2)^{1/2}$$

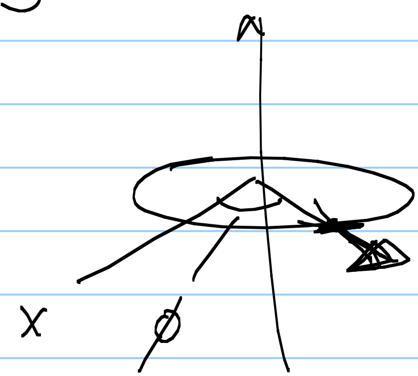
$$= \frac{\mu_0 I}{2\pi s}\,\hat{\phi}$$



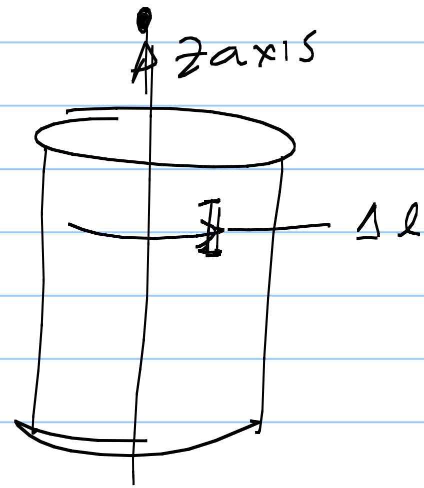
Ex (HW) Ring carrying I

Find B on the axis of ring

$$\int_0^{2\pi} \hat{s}\, d\phi = 0$$



Ex n turns/length, Current I



$$\vec{K} = \frac{I\, n \cdot \Delta l}{\Delta l}\,\hat{\phi}$$

$$B(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{k}(\vec{r}\,') \times (\bar{r} - \bar{r}\,')}{|r - r'|^3} \, ds'$$