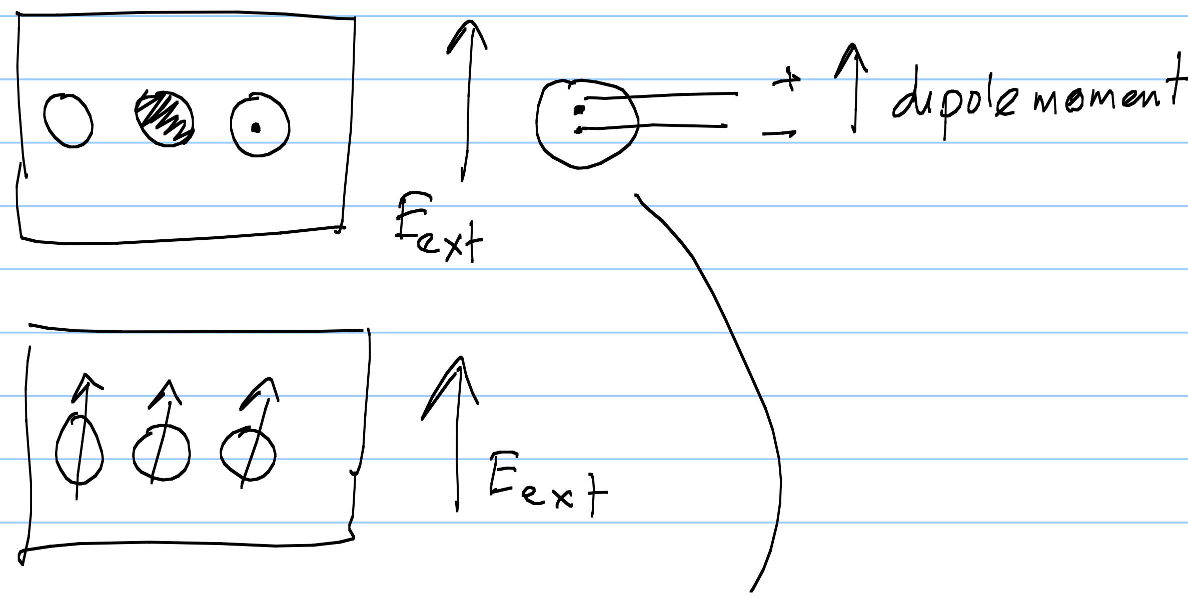
Dielectric Materials

Note Title 3/18/2009

- Conductors — Containers infinite charges (both kind)
- Insulators — Electrons tightly bound to atoms
  - Atomic/Molecular Properties.

Dominant Mechanisms

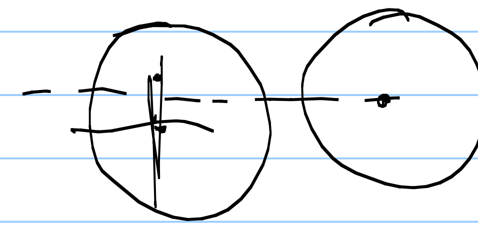
(i) Neutral Atoms



Empirical Rule $|E_{ext}|$ is small

$$\vec{p} = \alpha \vec{E}_{ext}$$

Atomic Polarizability

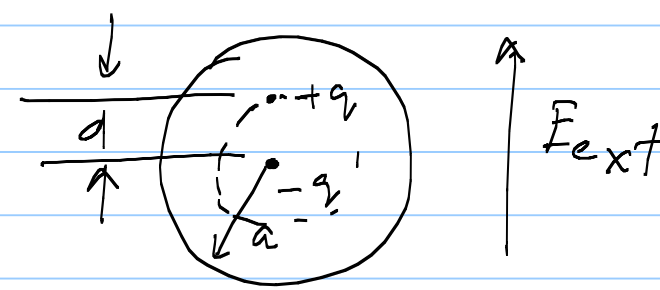


Crude model: Nucleus point charge q

Electron cloud rigid sphere uniform charge density

Force on nucleus

$$F = q\left(E_{ext} + E_e\right)$$



E_e : field of electron cloud

$$E_e = -\frac{q\,(d^3/a^3)}{4\pi\epsilon_0 d^2} = -\frac{qd}{4\pi\epsilon_0 a^3}$$

In equilibrium

$$q E_{ext} = \frac{q^2 d}{4\pi\epsilon_0 a^3}$$

Induced DM

$$p = qd = \underbrace{(4\pi\epsilon_0 a^3)}_{\alpha} E_{ext}$$

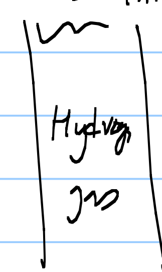
$$\frac{\alpha}{4\pi\epsilon_0} = a^3 \quad \text{in units } m^3$$

Accurate upto 25%

Example (4.1 G)

$$E = \frac{V}{D} = 5 \times 10^5 \ \frac{V}{m}$$

D = 1mm



V = 500 V

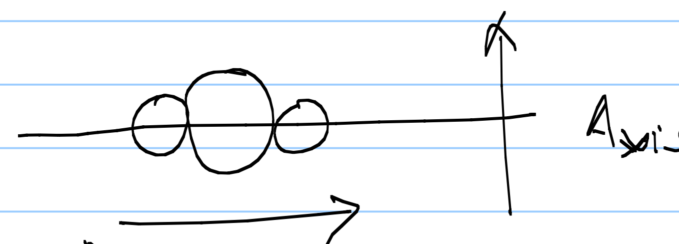
$\alpha_H = 0.667 \times 10^{-30}\ m^3 \quad m \quad (4\pi\epsilon_0)$

$p = ed = \alpha E_{ext}$

$d = \dfrac{\alpha E_{ext}}{e} \approx 2.3 \times 10^{-16}\ m \qquad \dfrac{d}{a} \approx 10^{-6}$

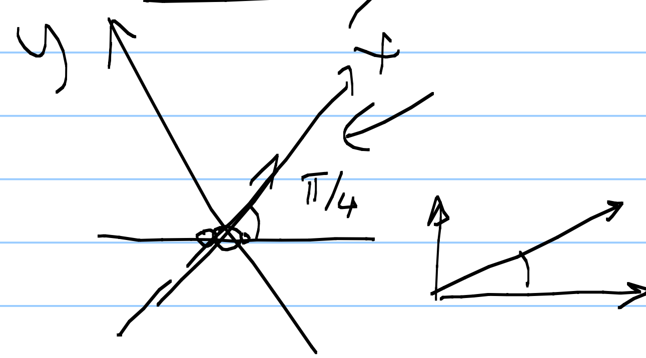
$\vec{p} = ed = 4 \times 10^{-35}\ C\text{-}m$

Molecules $\quad CO_2$



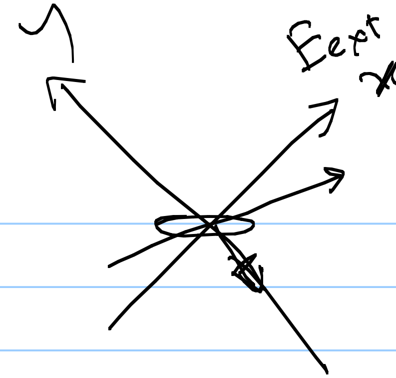
$\dfrac{\alpha_{\parallel}}{4\pi\epsilon_0} = 4.05 \times 10^{-30}\ m^3$

$\dfrac{\alpha_{\perp}}{4\pi\epsilon_0} = 1.75 \times 10^{-30}\ m^3$



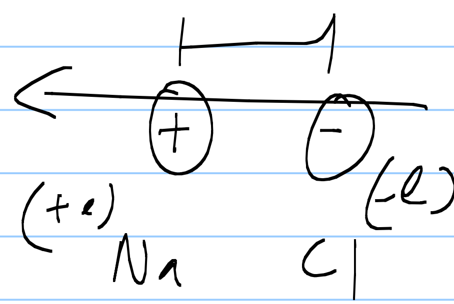
In general

$$\begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} = \underbrace{\begin{bmatrix} \alpha_{xx} & \alpha_{xy} & \alpha_{xz} \\ & \alpha_{yy} & \\ & & \alpha_{zz} \end{bmatrix}}_{\text{Polarizability Tensor of } 2^n \text{ rank}} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$



(ii) Polar Molecules:

Examples: NaCl



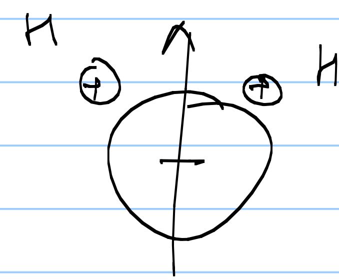
dipole moment

dipole moment $\vec{p}_{NaCl} = e \cdot \text{Bond length}$ $\qquad 2.36 \times 10^{-10}$ m

$= 3.77 \times 10^{-29}$ C-m

$\vec{p}_{nacl,exp} = 2.99 \times 10^{-29}$ C-m $\qquad$ 80% charge transfer

Example $\quad H_2O$

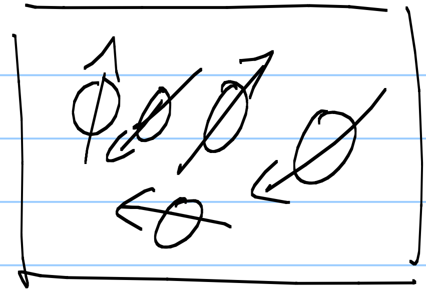


$\vec{p}_{H_2O} = 6.1 \times 10^{-30}$ C-m

NaCl → $\quad K = 6$ (Room temp)

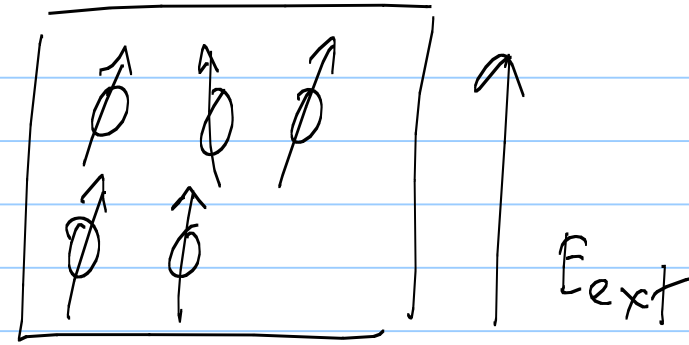
$H_2O \quad K = 80$ (25°C)

No E_{ext}



Net dipole moment $= 0$

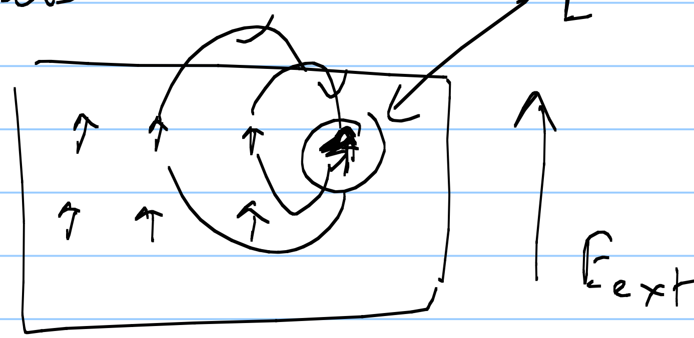
E_{ext}



$T \neq 0$

net dipole moment $\neq 0$

Dielectric Materials

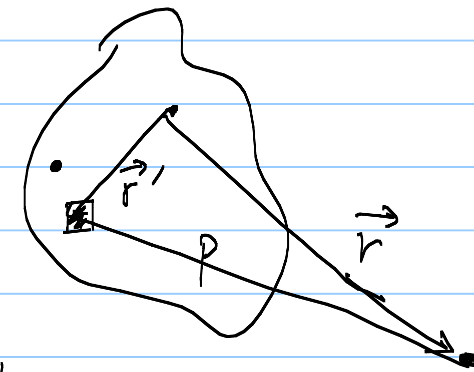


(i) Polarized material $\Longrightarrow$ Electric field

Potential due to a polarized object

$P(\vec{r}')$: Polarization at $\vec{r}'$

dv' at $\vec{r}'$, dipole moment $\sim dv'$



$$dp = P(\vec{r}')\,dv'$$

$$dV(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{d\bar{p}\cdot(\bar{r}-\vec{r}')}{|\bar{r}-\bar{r}'|^3}$$ $\rightarrow$ pure dipole

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\vec{P}(\vec{r}')\cdot(\vec{r}-\vec{r}')}{|r-r'|^3}\,dv'$$

$$\nabla\cdot(f\vec{A}) = f\,\nabla\cdot\bar{A} + \bar{A}\cdot\nabla f$$

$$\nabla' \cdot \left(\frac{\bar{P}(\vec{r}')}{|\bar{r}-\bar{r}'|} \right) = \frac{1}{|\bar{r}-\bar{r}'|} \bar{\nabla}' \cdot \bar{P} + \underbrace{\bar{P} \cdot \nabla' \left(\frac{1}{|\bar{r}-\bar{r}'|} \right)}$$

$$\nabla \frac{1}{r} = -\frac{\hat{r}}{r^2}$$

$$= \frac{\nabla' \cdot \bar{P}}{|\bar{r}-\bar{r}'|} + \vec{p} \cdot \frac{(\vec{r}-\vec{r}')}{|\bar{r}-\bar{r}'|^3}$$

$$\frac{d}{dx} \frac{1}{\sqrt{x^2+y^2+z^2}}$$

$$\left(-\frac{1}{2}\right) \frac{2x}{(x^2+y^2+z^2)^{3/2}}$$

$$V(\vec{r}) = \underbrace{\frac{1}{4\pi\epsilon_0} \int \nabla' \cdot \left(\frac{P(\vec{r}')}{|\bar{r}-\bar{r}'|} \right) dv'} + \underbrace{\frac{1}{4\pi\epsilon_0} \int \frac{(-\nabla' \cdot P)\, dv'}{|\bar{r}-\bar{r}'|}}$$

compare $\rho(\vec{r}')$

$$V = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(r')\, dv}{|r-r'|}$$

$$\frac{1}{4\pi\epsilon_0} \oint \frac{P(\vec{r}) \cdot \hat{n}\, ds}{|\bar{r}-\bar{r}'|}$$

$$\rho_b \rightarrow \frac{1}{4\pi\epsilon_0} \int \frac{\rho_b(\vec{r}')\, dv'}{|\bar{r}-\bar{r}'|}$$

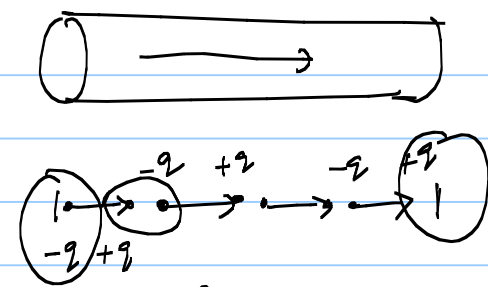
bound surface charge density

$$\sigma_b = P \cdot \hat{n}$$

bound volume charge density

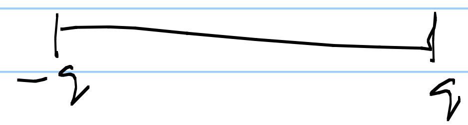
$$\rho_b = -\nabla' \cdot P$$

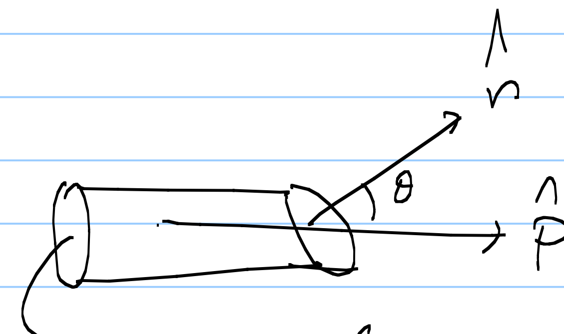
Interpretation



P uniform

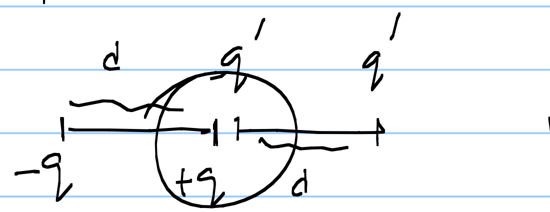
$\rho = 0$





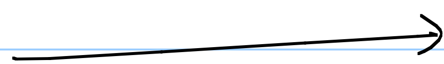
$\dfrac{-q}{A}$ $\qquad$ $\dfrac{q}{(A/\cos\theta)}$

Nonuniform P



$qd < q'd$

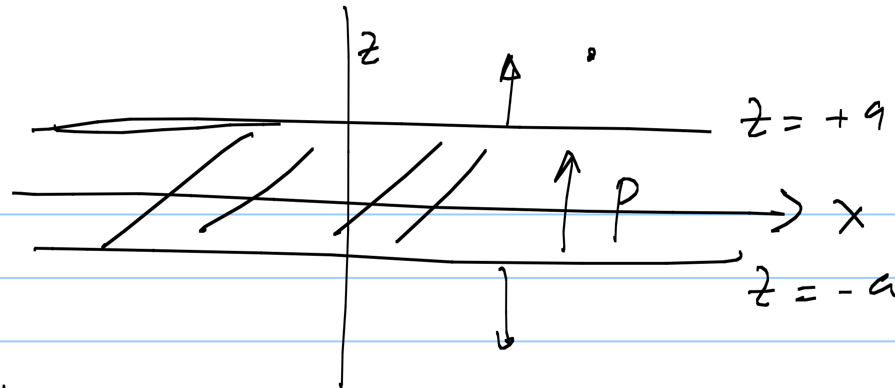
$q < q'$



P increasing

$\rho = -\nabla \cdot P$

Example



$\vec{P} = P_0 \hat{z}$

surface charge density $\sigma_b \Big|_{z=a} = P_0$

$\sigma_b \Big|_{z=-a} = -P_0$

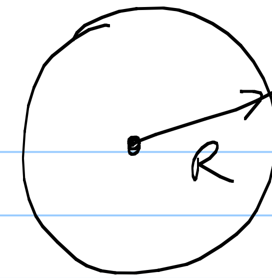
volume charge dens $\rho_b = 0$

$E(z) = \dfrac{P_0}{\epsilon_0}$ $\quad -a < z < a$

$= 0$ $\quad$ otherwise

Example: $P(\vec{r}) = k\vec{r}$ $\quad r < R$

$= 0$ $\quad r > R$



$$\rho_b = -\nabla\cdot P = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 k r\right) = -3k$$

$$\sigma_b\Big|_{r=R} = kR$$

Net change $= 0 = kR \cdot 4\pi R^2 + (-3k)\cdot \frac{4}{3}\pi R^3 = 0$

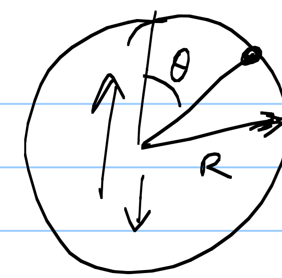
$$E(\vec{r}) = +\frac{\rho}{3\epsilon_0}\vec{r} = \frac{-k\vec{r}}{\epsilon_0} \qquad r < R$$

$$= 0 \qquad r > R$$

Ex Sphere with uniform Polarization

$$\vec{P}(\vec{r}) = P_0 \hat{z} \qquad r < R$$

$$= 0$$



$$\rho_b = 0$$

$$\sigma_b(R,\theta) = P_0 \cos\theta$$

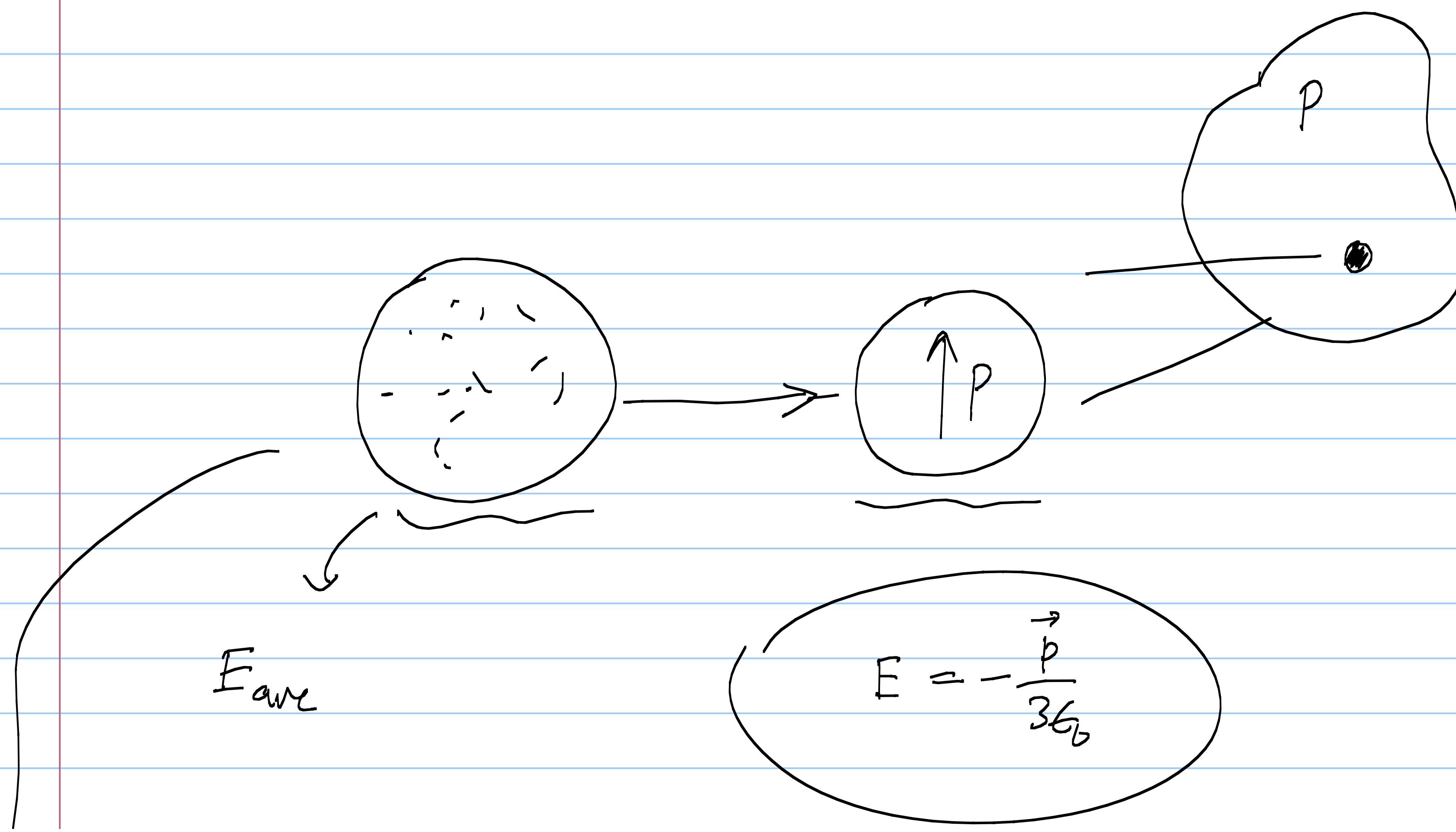
$$V(r,\theta) = \frac{P_0}{3\epsilon_0} r\cos\theta \qquad r < R \qquad \vec{E} = -\frac{P_0}{3\epsilon_0}\hat{z}$$

$$= \frac{P_0 R^3}{3\epsilon_0} \frac{\cos\theta}{r^2}$$

Electric fields

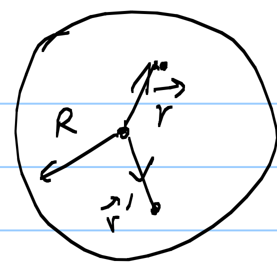
→ In vacuum   Actual EF

→ Materials   Averaged Fields



$$E = -\frac{\vec{p}}{3\epsilon_0}$$

q at $\vec{r}$



$$E_{ave} = \frac{1}{\frac{4}{3}\pi R^3}\int E(\vec{r}\,')\,dv'$$

$$E(\vec{r}\,') = \frac{1}{4\pi\epsilon_0}\,\frac{q(\vec{r}\,'-\vec{r})}{|\vec{r}\,'-\vec{r}|^3}$$

ρ unifm

$$E_{ave} = \frac{(q/V)}{4\pi\epsilon_0}\int\frac{(\vec{r}\,'-\vec{r})}{|\vec{r}\,'-\vec{r}|^3}\,dv'$$

$$= \frac{1}{4\pi\epsilon_0}\int\frac{(-q/V)(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\,dv'$$

= Electric field at $\vec{r}$ due ~~to~~ a uniformly change sphere den $(-q/V)$

$$E = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\vec{r}\,')\,dv'(r-r')}{|r-r'|^3}$$

$$= \frac{\rho \vec{r}}{3\epsilon_0} = \frac{-q\vec{r}}{\frac{4\pi}{3}R^3 \cdot 3\epsilon_0}$$

$$= \frac{-q\vec{r}}{4\pi\epsilon_0 R^3}$$

$$= \frac{-\text{net dipole moment}}{4\pi\epsilon_0 R^3}$$

$$E_{ave} = \frac{-\text{net dipole moment}}{4\pi\epsilon_0 R^3}$$

$$= -\frac{\vec{p}_n / \left(\frac{4}{3}\pi R^3\right)}{3\epsilon_0}$$

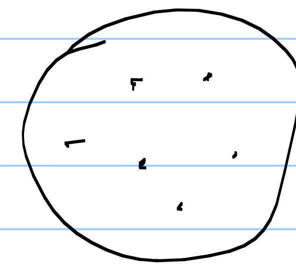
$$= -\frac{\vec{P}}{3\epsilon_0}$$

$\vec{P}$ ~~P~~ Polarization

Gauss Law For Dielectrics

Bound charges : ρ_b

Free charge : ρ_f



$$\rho = \rho_b + \rho_f$$

$$\epsilon_0 \nabla \cdot E = -\nabla \cdot P + \rho_f$$

$$\Rightarrow \quad \rho_f = \nabla \cdot \underbrace{(\epsilon_0 E + P)}_{\text{Electric displacement field}}$$

$$\boxed{\nabla \cdot D = \rho_f} \qquad D = \epsilon_0 E + P$$

$$\oint D \cdot \hat{n} \, ds = Q_{f,\text{enclosed}}$$