Multipole Expansion

Note Title 2/25/2009

Legendre Polynomials

$$GF = G(t,x) = \frac{1}{(1+t^2-2tx)^{1/2}} \qquad -1 < t, x < +1$$

$$= \frac{1}{(1+\epsilon)^{1/2}} \qquad \epsilon = t(t-2x)$$

$|t| < 1$

$$= 1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 - \frac{5}{16}\epsilon^3 + \cdots \quad + \frac{(2n-1)!!}{2^n n!}(-1)^n \epsilon^n$$

$$(2n-1)!! = (2n-1)(2n-3) \cdots 1$$

$$= 1 - \frac{1}{2}t(t-2x) + \frac{3}{8}t^2(t-2x)^2 - \frac{5}{16} \cdots$$

$$= 1 + (\underbrace{x}) t + (\underbrace{-\tfrac{1}{2} + \tfrac{3}{2}x^2}) t^2 + (\underbrace{-\tfrac{3x}{2} + \tfrac{5x^3}{2}}) t^3 + \cdots$$

$$= \sum_{n=0}^{\infty} t^n P_n(x)$$

$P_n(x)$: Legendre Polynomials

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$P_2(x) = (3x^2 - 1)/2$$
$$P_3(x) = (5x^3 - 3x)/2$$
$$P_4(x) = (35x^4 - 30x^2 + 3)/8$$

Properties

(i) $-1 \le P_n(x) \le 1 \quad \forall\, n, x$

(ii) $P_n(-x) = P_n(x)$ n even

$\qquad\qquad = -P_n(x)$ n odd

(iii) Rodrigues Formula

$$P_n(x) = \frac{1}{2^n\, n!}\left(\frac{d}{dx}\right)^n (x^2-1)^n$$

(iv) Orthogonality

$$\int_{-1}^{1} P_n(x)\, P_m(x)\, dx = \frac{2}{2n+1} \qquad \text{if } m = n$$

$$= 0 \qquad \text{if } m \neq n$$

(v) Complete

$$F : [-1,1] \to \mathbb{R}$$

$$F(x) = \sum_{n=0}^{\infty} c_n P_n(x) \checkmark$$

$$\boxed{\int_0^{2\pi} \sin(mx)\sin(nx)\, dx = 0 \quad \text{if } m \neq n}$$

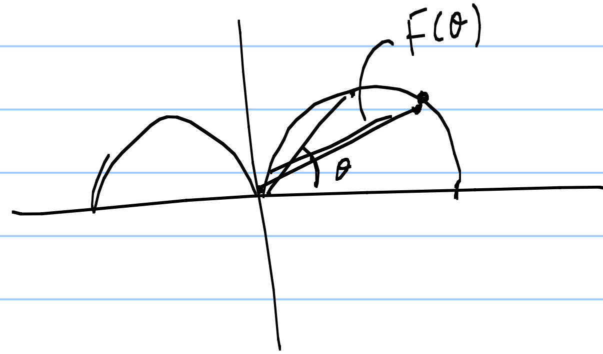
$$\int_{-1}^{1} x^m x^n \, dx$$

(iv) $P_n[\cos\theta]$

$x = \cos\theta$

$x \in [-1, 1]$

$\theta \in [0, \pi]$



$F(\theta) = \cos\theta$ $\quad \theta : 0, 2\pi$

$(r = F(\theta), \theta)$

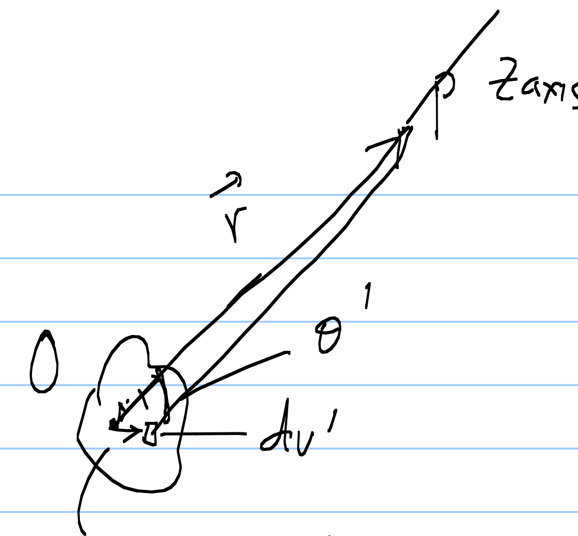
$(1, 0)$

$\left(\frac{\sqrt{3}}{2}, \frac{\pi}{6}\right)$ $\quad \left(\frac{1}{\sqrt{2}}, \frac{\pi}{4}\right)$

$(0, \pi/2)$

Multipole Expansion

Let ρ : Localized charge distribution



$$V(P) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}\,')\,dv'}{|\bar{r}-\bar{r}'|}$$

$\vec{r}\,' = (r', \theta', \phi')$

$$\frac{1}{|\bar{r}-\bar{r}'|} = \frac{1}{(r^2+r'^2-2rr'\cos\theta')^{1/2}}$$

$x = \cos\theta'$ put $t = r'/r$

$t < 1$

$$\frac{r'}{r}\left(\frac{r'}{r}-2\cos\theta'\right) < 1$$

$$= \frac{1}{r(1+t^2-2tx)^{1/2}}$$

$\epsilon < 1$

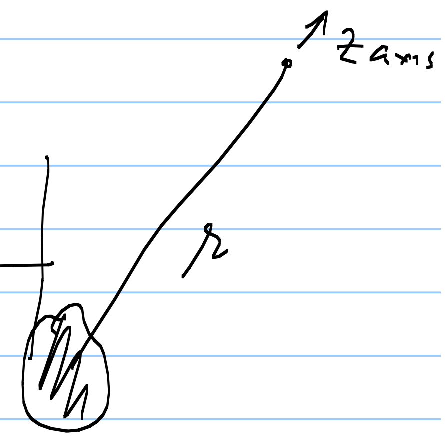
$$= \frac{1}{r}\sum_n \left(\frac{r'}{r}\right)^n P_n(\cos\theta')$$

$$V(P) = \frac{1}{4\pi\epsilon_0}\left[\sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \int \rho(\vec{r}\,')\, r'^n\, P_n(\cos\theta')\, dv'\right]$$

$$= \frac{1}{4\pi\epsilon_0}\Bigg[\underbrace{\frac{1}{r}\int \rho(\vec{r}\,')\, dv'}_{\text{monopole term}} + \underbrace{\frac{1}{r^2}\int r'\cos\theta'\, \rho(\vec{r}\,')\, dv'}_{\text{dipole term}} + \underbrace{\frac{1}{r^3}\int r'^2\left(\frac{3\cos^2\theta' - 1}{2}\right)\rho(\vec{r}\,')\, dv'}_{\text{Quadrupole term}} + \cdots\Bigg]$$

Multipole Expansion

$r \gg r'$



$$\left|\frac{n^{th} \text{ term}}{(n-1)^{th} \text{ term}}\right| = \left|\frac{\frac{1}{r^{n+1}}\int r'^n \rho(\vec{r}\,')\, P_n(\cos\theta')\, dv'}{\frac{1}{r^n}\int r'^{n-1}\rho(\vec{r}\,')\, P_{n-1}(\cos\theta')\, dv'}\right|$$

$$\sim \frac{r'}{r} \sim 10^{-5}$$

$r' \sim 10^{-15}$ m

$r \sim 10^{-10}$ m

First term = $\int \rho(\vec{r}\,')\,dv' = Q_{net}$ = Monopole moment

Dipole term $\int r' \cos\theta' \rho(\vec{r}\,')\,dv'$

$r'\cos\theta' = \vec{r}\,' \cdot \hat{r}$

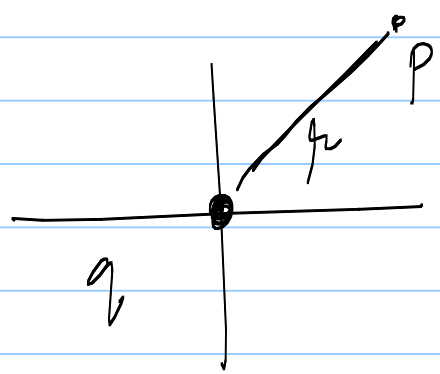
$$\hat{r} \cdot \underbrace{\int \vec{r}\,' \rho(r')\,dv'}_{\vec{p}}$$

$\vec{p}$: dipole moment

$$\frac{\vec{p}\cdot\hat{r}}{4\pi\epsilon_0 r^2}$$

Example

(i)

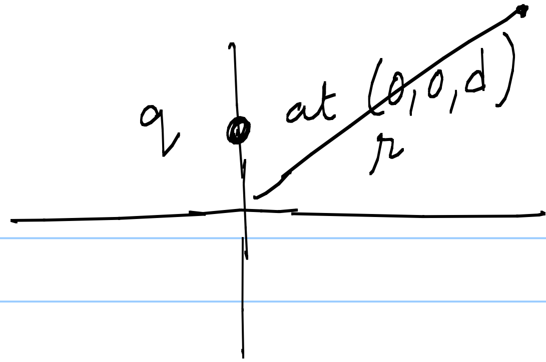


Monopole moment $= q$

Dipole moment $= 0$

$$V(P) = \frac{1}{4\pi\epsilon_0}\frac{q}{r} + 0 - $$

(ii)



Monopole moment : q

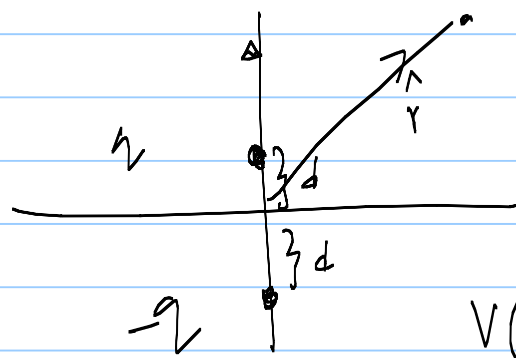
dipole moment : $qd\hat{z}$

$\sum_i \vec{r}_i' q_i$

if $r \gg d$

$\rightarrow$ $$V(P) = \frac{q}{4\pi\epsilon_0 r} + \frac{qd\hat{z}\cdot\hat{r}}{4\pi\epsilon_0 r^2} + \cdots$$

(iii)

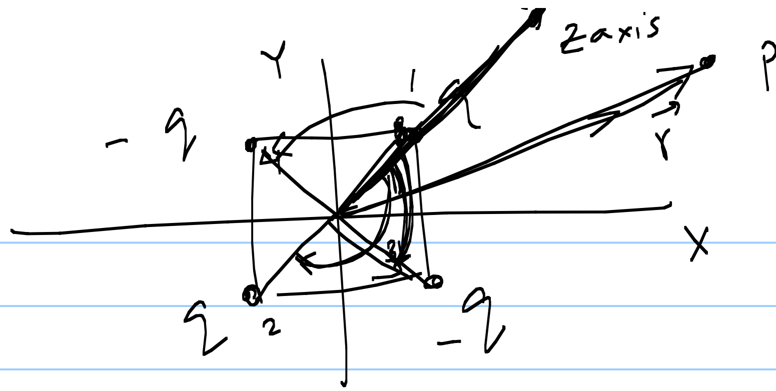


Monopole moment $= 0$

Dipole moment $= 2qd\hat{z}$

$$V(P) = 0 + \frac{2qd\hat{z}\cdot\hat{r}}{4\pi\epsilon_0 r^2} + \cdots$$

(iv)



Monopole moment $= 0$

Dipole momen $= 0$

Quadrupole moment $=$

$$P \equiv (x, x, 0)$$

q at $\pm(d, d, 0)$

$-q$ at $\pm(d, -d, 0)$

$$\frac{1}{4\pi\epsilon_0 r^3} \sum_i r_i^2 \frac{(3\cos^2\theta_i - 1)}{2} q_i$$

$$V(P) = 0 + 0$$

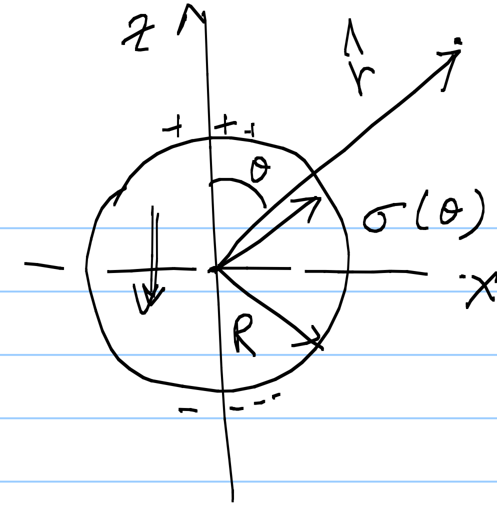
$$\frac{1}{4\pi\epsilon_0 r^3}\Big[(\sqrt{2}d)^2 \frac{(3\cdot 1 - 1)}{2} q + (\sqrt{2}d)^2 \frac{(3\cdot 1 - 1)}{2}$$

$$+ (\sqrt{2}d)^2 \frac{(3\cdot 0 - 1)}{2}(-q) + (\sqrt{2}d)^2 \Big(\frac{3\cdot 0 - 1}{2} - q\Big)\Big]$$

$$= \frac{6qd^2}{4\pi\epsilon_0 (\sqrt{2}x)^3}$$

Ex Spherical Shell Radius R

$$\sigma(\theta) = P\cos\theta$$



Find approximate Potential.

$$\text{Monopole moment} = \int \sigma(\theta')\,ds' = 0$$

$$= PR^2 \underbrace{\int_0^{\pi} \cos\theta' \sin\theta'\, d\theta'}_{0} \int_0^{2\pi} d\phi' = 0$$

$$\text{Dipole moment} = \int \vec{r}\,'\,\sigma(\theta')\,ds'$$

$$= \int \left(R\sin\theta' \cos\phi'\,\hat{x} + R\sin\theta' \sin\phi'\,\hat{y} + R\cos\theta'\,\hat{z}\right)$$

$$\times \left(P\cos\theta'\right)\left(R^2 \sin\theta'\, d\theta'\, d\phi'\right)$$

$$= PR^3\hat{z} \int_0^{\pi} \cos^2\theta' \sin\theta'\, d\theta' \int_0^{2\pi} d\phi'$$

$$= \frac{pR^3}{3}(4\pi)\,\hat{z}$$

Dipole term of $V(P) = \dfrac{1}{4\pi\epsilon_0 r^2}\left(\dfrac{pR^3\,4\pi}{3}\,\hat{z}\right)\cdot\hat{r}$

$P \equiv (r, \theta, \phi)$

$$= \frac{pR^3\cos\theta}{3\epsilon_0 r^2}.$$

$$\vec{E}(P) = -\nabla V = \frac{pR^3}{3\epsilon_0}\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right]$$

$$\nabla \equiv \hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial \phi}$$

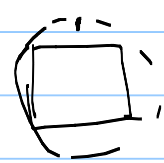
$V(P) = \dfrac{pR^3\cos\theta}{3\epsilon_0 r^2}$ $\leftarrow$ Exact Formula

behaves like pure dipole

$r > R$

pure "point" dipole

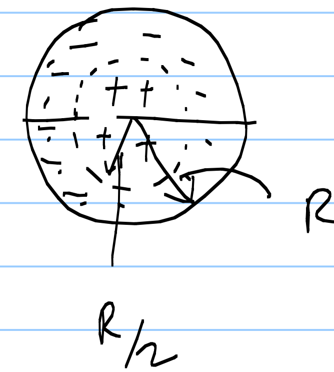
$r > R$



Ex Volume charge density

$$\underline{\rho(r,\theta) = k\,\frac{R(R-2r)}{r^2}\,\sin\theta}$$

Find Approx Potential on z axis

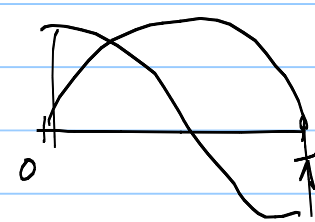


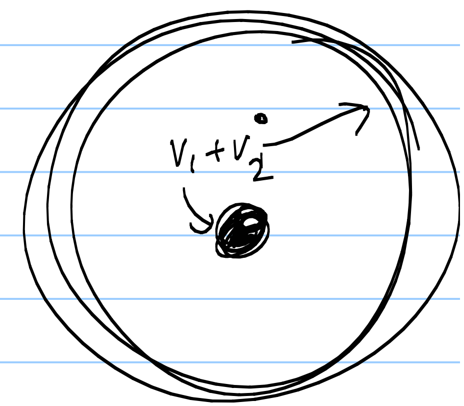
Monopole Moment = 0

Dipole Moment

$$= \int \underline{\underline{\vec{r}\,'}}\; k\,\frac{R(R-2r')}{\cancel{r'^2}}\,\sin\theta'\,\cancel{r'^2}\sin\theta'\,d\theta'\,d\phi'\,dr'$$

$$= kR\int r'(R-2r')\,dr' \int \underbrace{\cos\theta'\sin^2\theta'\,d\theta'}_{0}\;\hat{z}\,(2\pi)$$





Quadrupole term =

$$\frac{1}{4\pi\epsilon_0 z^3} \int r^2 \left(\frac{3\cos^2\theta - 1}{2}\right) \frac{kR(R-2r)}{\cancel{r^2}} \sin\theta \, \cancel{r^2} \sin\theta \, d\theta \, d\phi \, dr$$

HW

$$\left(r'/r\right)$$

$$1 + \frac{r'}{r} + \left(\frac{r'}{r}\right)^2 + \cdots$$

$$V_{Quadrupole} = \frac{1}{4\pi\epsilon_0 r^3} \int r'^2 \, \frac{(3\cos^2\theta' - 1)}{2} \, \rho(\vec{r}') \, dv'$$

$\vec{r} = (x, y, z)$

$\vec{r}' = (x', y', z')$

$$\cos\theta' = \frac{\vec{r} \cdot \vec{r}'}{|\vec{r}|\,|\vec{r}'|}$$

$$= \frac{(x x' + y y' + z z')}{\underline{(x^2+y^2+z^2)}^{1/2}\,\underline{(x'^2+y'^2+z'^2)}^{1/2}}$$

$$\underline{3\cos^2\theta' - 1} = \left[3(x x' + y y' + z z')^2 - (x^2+y^2+z^2)\,r'^2\right] / r^2 r'^2$$

$$= \left[x^2(3x'^2 - r'^2) + y^2(3y'^2 - r'^2) + z^2(3z'^2 - r'^2)\right.$$

$$xy(3x'y') + yx(3x'y') \quad 2xy(3x'y') + 2xz(3x'z') + 2yz(3y'z')\Big]$$

$$= [x,\ y,\ z] \begin{bmatrix} 3x'^2 - r'^2 & 3x'y' & 3x'z' \\ 3x'y' & 3y'^2 - r'^2 & \\ & & \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \Big/ r^2 r'^2$$

$$V_{quad} = \frac{1}{4\pi\epsilon_0 r^3} \int r'^2 \, \frac{3\cos^2\theta' - 1}{2} \, \rho(\vec{r}') \, dv'$$

$$= \frac{1}{4\pi\epsilon_0 r^5} [x, y, z] \begin{bmatrix} \underbrace{\int (3x'^2 - r'^2) \rho(\vec{r}') \, dv'}_{Q_{xx}} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \frac{1}{2}$$

$$Q_{xx} = \int \left(3x'^2 - r'^2\right) \rho(\vec{r}')\, dv'$$

Q Tensor of rank 2