Note Title 2/19/2009

Separation of variables

$$\left(\frac{d^2}{dx^2} + w^2\right) y = 0$$

Example 1: $\frac{d^2y}{dx^2} + w^2 y = 0$ $\quad x \in [0, \pi]$

$y = A \sin(wx)$ $\quad \cos(wx)$ $\quad e^{iwx}$

$$y = A\sin(wx + B) = \boxed{A' \sin(wx) + B' \cos(wx)}$$

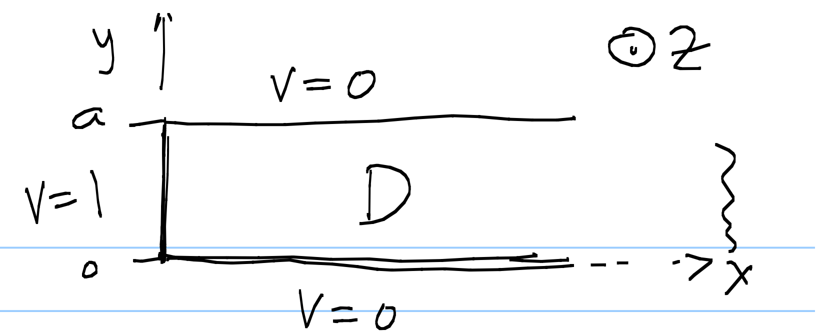
$$= A' \sin(wx) + B' \cos(wx) + C' e^{iwx} + D' e^{-iwx}$$

$y_1 = \sin(wx)$

$y_2 = \cos(wx)$ $\quad y_1 \neq c y_2$

$y_3 = e^{iwx}$ $\quad \{y_1, y_2, y_3\}$ not ~~Linear~~ ind

Example 2:



D: $0 < x < \infty$

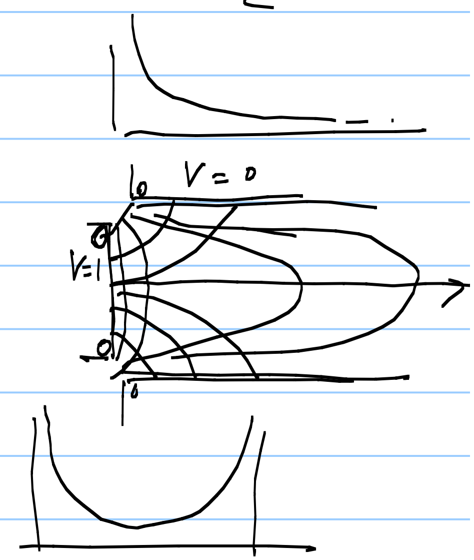
$0 < y < a$

To solve

$$\nabla^2 V = 0$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) V = 0$$

pot V independent of z



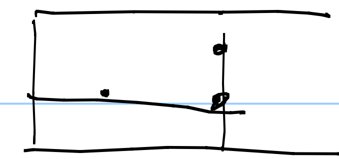
BC: (i) $V(x,0) = 0$

(ii) $V(x,a) = 0$ $\Big\}$ $0 < x < \infty$

(iii) $V(0,y) = V_0(y)$

(iv) $V(x \to \infty, y) = 0$ $\Big\}$ $0 < y < a$

$$f(x,y) = X(x)\,Y(y)$$



$$\nabla^2 f = 0$$

$$\frac{\partial^2}{\partial x^2} X Y + \frac{\partial^2}{\partial y^2} X Y = 0$$

$$\Rightarrow \frac{1}{X(x)} \frac{\partial^2}{\partial x^2} X(x) = -\frac{1}{Y(y)} \frac{\partial^2}{\partial y^2} Y(y) = G \ge \text{Const} \quad \forall \text{ points in } D$$

$$= k^2$$

$$\frac{d^2}{dx^2} X(x) = k^2 X(x) \Rightarrow X_k(x) = C_k e^{kx} + D_k e^{-kx}$$

$$\frac{d^2}{dy^2} Y(y) = -k^2 Y(y) \Rightarrow Y_k(y) = A_k \sin(ky) + B_k \cos(ky)$$

For each k $\quad f_k(x,y) = \left(C_k e^{kx} + D_k e^{-kx} \right)\left(A_k \sin(ky) + B_k \cos(ky) \right)$

$$V(x,y) = \sum_k f_k(x,y)$$

(i) $V(x \to \infty, y) = 0 \Rightarrow f_k(x \to \infty, y) = 0$

$$\lim_{x \to \infty} \left(C_k e^{kx} + \cancel{D_k e^{-kx}}^{\to 0} \right) \Rightarrow C_k = 0 \quad \forall k.$$

(ii) $V(x,0) = 0 \Rightarrow f_k(x,0) = 0$

$$(x \ldots \text{part})(A_k \cdot 0 + B_k \cdot 1) = 0 \Rightarrow B_k = 0$$

$$f_k : \; e^{-kx} \sin(ky)$$

(iii) $V(x,a) = 0 \quad f_k(x,a) = 0$

$$(x \ldots \text{part})(A_k \sin(ka)) = 0 \Rightarrow k = \frac{n\pi}{a} \quad n = 1, 2, \ldots$$

discard $n = 0, -1, \ldots$

$\underline{\sin(-ky)} = \underbrace{-}_{(-1)}\sin(ky)$

Relabel

$$f_k = T_k e^{-kx} \sin(ky)$$

$$f_n = T_n \exp\left(-\frac{n\pi x}{a}\right) \sin\left(\frac{n\pi y}{a}\right)$$

(iv) $V(0,y) = V_0(y)$

$$\boxed{V = \sum_{n=1}^{\infty} T_n \exp\left(-\frac{n\pi x}{a}\right) \sin\left(\frac{n\pi y}{a}\right)}$$

Apply BC to V

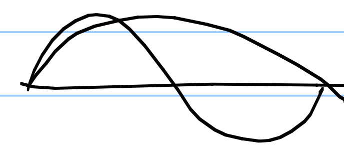
$$V_0(y) = \sum_{n=1}^{\infty} T_n \sin\left(\frac{n\pi y}{a}\right)$$

<u>Fourier Series</u>

$$V_0(y) \sin\left(\frac{m\pi y}{a}\right) = \sum_n T_n \sin\left(\frac{n\pi y}{a}\right) \sin\left(\frac{m\pi y}{a}\right)$$

$$\int_0^a V_0(y)\sin\left(\frac{m\pi y}{a}\right)dy = \sum_n T_n \underbrace{\int_0^a \sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{m\pi y}{a}\right)dy}$$

$m \neq n$ $\quad = 0$



when

$m = n$ $\quad \frac{a}{2}$

$$= T_m \cdot \frac{a}{2}$$

$$T_m = \frac{2}{a}\int_0^a V_0(y)\sin\left(\frac{m\pi y}{a}\right)dy$$

$$V_0(y) = V_0 \qquad \forall\, y \in [0, a]$$

$$T_n = \frac{2V_0}{a} \int_0^a \sin\left(\frac{n\pi y}{a}\right) dy$$

$$= \frac{2V_0}{a} \cdot \frac{a}{\pi n} \cdot \left[1 - \cos(n\pi)\right]$$

$$= \frac{4V_0}{n\pi} \qquad \text{if } n \text{ is odd}$$

$$= 0 \qquad \text{if } n \text{ is even}$$

$$V(x,y) = \sum_{n=1,3,\ldots}^{\infty} \left(\frac{4V_0}{n\pi}\right) \cdot \exp\left(-\frac{n\pi x}{a}\right) \sin\left(\frac{n\pi y}{a}\right)$$

$$= \frac{2V_0}{\pi} \tan^{-1}\left(\frac{\sin\left(\frac{\pi y}{a}\right)}{\sinh\left(\frac{\pi x}{a}\right)} \right)$$