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Towards Holonomy Decomposition of Process Algebras

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Abstract

This work aims at understanding the structure of process algebras via holonomy decomposition. In that connection, the work studies the skeleton of the transformation semigroup which is obtained from the natural transition relation between the processes of process algebra, and observes that it is of height one.

Keywords: *Process algebra, Transformation semigroup, Holonomy decomposition.*

1. INTRODUCTION

Process algebras represent a mathematically rigorous framework for modeling concurrent systems of interacting processes. The process algebraic approach relies on equational and inequational reasoning as the basis for analyzing the behavior of such systems. Although Calculus of Communicating Systems (CCS) by Milner [6] is the seminal work towards algebraic process theory, the term 'process algebra' was coined by Bergstra and Klop [1] in their Algebra of Communicating Processes (ACP). Process algebra is a structure in the sense of universal algebra that satisfies a particular set of axioms. In fact, ACP is an equational reformulation of CCS. Since then, the phrase 'process algebra' is also used to refer an area of science which studies concurrent processes in algebraic approach.

In this work, we consider the process algebra, viz. basic process algebra (BPA), which is the kernel of all other process theories of ACP-style, to study its structure using holonomy decomposition. Eilenberg's holonomy decomposition theorem for transformation semigroups [2] is a sophisticated version of Krohn-Rhodes decomposition theorem for finite automata [5]. The Krohn-Rhodes prime decomposition theorem determines the building blocks and their combination so as to mimic any given finite automaton. The theorem, indeed, states that every finite automaton can be simulated by a cascade of basic types of irreducible automata, viz. flip-flops and automata whose syntactic monoids are finite simple groups dividing the syntactic monoid of the given finite automaton. The theorem is, in general, applicable to labeled transition systems; so also the holonomy decomposition theorem. Utilizing this feature, using holonomy decomposition theorem of transformation semigroups, Holcombe [3] and Krishna and Chatterjee [4] have studied the structures of near-rings and seminearrings, respectively. In both the cases, they use the transition system induced by the binary operation multiplication of the respective algebras. As BPAs are special cases of seminearrings, one may customize the work of [4] to process algebras. But the transition system induced by multiplication of BPA might not be interesting for all practical purposes of process algebras; rather the natural transition relation (refer para 2 of Section 3, below) between the processes is very essential for process algebras. Thus, using holonomy decomposition for the transition system induced by the natural transition relation between the processes, we aim to study the structure of process algebras.

In order to study the structure of an algebra using holonomy decomposition theorem, primarily, there are three steps to pursue (see Theorem 2.1, below). 1. Construction of transformation semigroup over the algebra, with respect to the desired transition relation. 2. Examining the skeleton of the transformation semigroup by investigating its equivalence classes and height. 3. Determining the respective holonomy groups for each equivalence class of the skeleton. In this paper, we pursue the first two steps for holonomy decomposition of process algebras in the direction of understanding their structure.

2. PRELIMINARIES

In this section we explore some fundamentals related to BPAs and holonomy decomposition of transformation semigroups which shall be useful in this work. For more details one may refer [1, 2].

2.1 Process algebra

In this subsection, we give a systematic introduction to the process algebra under consideration, namely, *basic process algebra* (BPA). Any other process algebra of ACP-style is a conservative extension of BPA, with additional features. Along with a set of atomic actions, BPA considers two fundamental (binary) operations between processes: the *sequential composition* $\bullet$ ($x \bullet y$, simply denoted by xy, is the process that executes x, and upon completion of x starts y) and the *alternative composition* + ($x + y$ is the process that either executes x or executes y).

Thus, formally, the signature of BPA is $\sum = \{\bullet\ ,\ +\} \cup A$, where A has a number of constants (for atomic actions); and the equational specification of BPA is $(\sum,\ E)$, where E consists of the following equations:

$$x + y = y + x$$
$$(x + y) + z = x + (y + z)$$
$$x + x = x$$
$$(xy)z = x(yz)$$
$$(x + y)z = xz + yz$$

If P is a model for BPA, the elements of its domain are called *processes*. Two important additional constants δ and ε shall be added to the signature of BPA to add the features of *deadlock* (a process which cannot do anything; neither execute an action, nor terminate successfully) and *empty-process* (a process that terminates immediately), respectively. The equational specification of BPA with deadlock, denoted by BPA_δ has the additional constant δ and the following two additional equations:

$$x + \delta = x \quad \text{and} \quad \delta x = \delta.$$

Also, the equational specification of BPA with ε, denoted by BPA^ε, has the additional constant ε and the following two additional axioms:

$$x\varepsilon = x = \varepsilon x.$$

The initial algebra of BPA (or BPA_δ or BPA^ε or $\text{BPA}^\varepsilon{}_\delta$) consists of equivalence classes of closed BPA (or BPA_δ or BPA^ε or $\text{BPA}^\varepsilon{}_\delta$) terms, where two terms are equivalent if they can be proved equal using the (respective) axiom system. The initial algebra is a model in bisimulation semantics and it is known that the axiom systems are complete axiomatization of bisimulation semantics.

Another important semantics is trace semantics. A *trace* of a process is a sequence of atomic actions (i.e. a word in A^*, the free monoid over the set of atomic actions A) that can be performed. In trace semantics two processes are equal if they have the same set of traces. The axiom system of BPA will have the additional equation

$$x(y + z) = xy + xz$$

so that it is a complete axiomatization of trace semantics. In this work, we will consider BPA with respect to trace semantics.

2.2 Holonomy Decomposition

A pair (P, S) with a nonempty set P and a semigroup S is called a *transformation semigroup* if there is an embedding $\varphi : S \to \mathcal{M}(P)$, where $\mathcal{M}(P)$ is the monoid of all partial functions on P with respect to composition. Let us denote the action of $s \in S$ on $p \in P$ as ps, rather than $p\varphi(s)$. For $p \in P$, let $\overline{p}$ be the constant function on P which takes the value p, i.e. $\overline{p}(q) = p, \forall q \in P.$ The closure of a transformation semigroup (P, S) is defined as $\overline{(P,S)} = (P,\overline{S}),$ where $\overline{S}$ is the monoid generated by $S \cup \bigcup_{p \in P} \{\overline{p}\}.$ Associated with (P, S), $\Im^0$ is the space of all s-images of P, where s-image of P, denoted by Ps, is $\{ps \mid p \in P\}$. The *skeleton space* $\Im$ of (P, S) is $\Im^0 \cup \{P\} \cup \bigcup_{p \in P} \{\{p\}\}$ with preorder $\leq$ defined by: for $A, B \in \Im$, $A \leq B$ if and only if $A \subseteq Bs$ for some $s \in S$. Define an equivalence relation $\sim$ on $\Im$ by putting $A \sim B$ if and only if $A \leq B$ and $B \leq A$. For $A \in \Im^0$ write $K(A)$ to denote the set of elements of S that behave as units on A, i.e.

$$K(A) = \left\{ f \in S \,\middle|\, \exists g \in S \text{ with } f(A) = A \text{ and } fg(a) = gf(a) = a, \forall a \in A \right\}.$$

Define *paving* of *A*, denoted by *B*(*A*), to be the set of maximal elements (with respect to set inclusion) of $\Im$ that are contained in *A*, i.e.

$$B(A) = \left\{ B \in \Im \,\middle|\, B \subseteq A \text{ and if } C \in \Im \text{ with } B \subseteq C \subseteq A \text{ then } C = B \text{ or } C = A \right\}.$$

Each $s \in K(A)$ acts as a permutation on *B*(*A*) and the set $\mathcal{G}(A)$ of the distinct permutations *B*(*A*) induced by the elements of *K*(*A*) is called holonomy group of *A*. The group $\mathcal{G}(A)$ acts as a transformation group on the paving of *A*.

Due to Eilenberg, any transformation semigroup of finite height can be covered by wreath product of holonomy transformation groups [2]. More precisely,

Theorem 2.1 (Holonomy decomposition of transformation semigroups)**.** *If* (*P, S*) *is a transformation semigroup of finite height and* $h : \Im \to \mathbb{Z}$ *is a height function* (*refer Definition 3.3, below*) *then*

$$(P,S) \prec \overline{\mathcal{H}_n} \circ \overline{\mathcal{H}_{n-1}} \circ \cdots \circ \overline{\mathcal{H}_1}$$

where $n = h(P)$ *and*

$$\mathcal{H}_i = \left(\prod_{j \in J} B(A_{ij}), \prod_{j \in J} \mathcal{G}(A_{ij}) \right)$$

in which $\left\{ A_{ij} \,\middle|\, j \in J \right\}$ *is the set of representatives of equivalence classes (with respect to* ~ *) in* $\Im(i)$, *the set of all elements of* $\Im$ *of height i.*

3. TRANSFORMATION SEMIGROUP OF PROCESS ALGEBRA AND ITS SKELETON

Let *A* be a set of atomic actions and *P* be the respective BPA over *A* with respect to trace semantics. Then, since • distributes over + from both sides, one may easily observe that the canonical representation of each process in *P* is a finite sum of distinct elements of A^*, the set of all words over *A*. Thus,

$$P = \left\{ \sum_{k=1}^{n} x_{i_k} \,\middle|\, x_{i_k} \in A^* \text{ and for } 1 \le j < k \le n, x_{i_j} \ne x_{i_k} \right\}.$$

The natural transition relation between the processes is the so-called *action relation*, which can be described as follows: For $a \in A$ and $p, p' \in P$, the action relation $p \xrightarrow{a} p'$ which denotes that *p* can execute *a* and there by turn into *p*', called *a* acts on *p* to yield *p*'. Formally,

$$p \xrightarrow{a} p' \Leftrightarrow ap' + q = p, \text{ for some } q \in P.$$

We now extend this action relation over all processes of *P* and customize to the present context to from a transformation semigroup over *P*. For $p, q \in P$, we define the action relation $p \xrightarrow{q} p'$ in which *q* acts completely (to the possible extent) on *p* and yields a process *p*', i.e. when *qp*' + *r* = *p*, for some $r \in P$, then *r* cannot be a summand of *qs*, for all $s \in P$. Formally,

$$p \xrightarrow{q} p' \Leftrightarrow \exists r \in P \text{ such that } qp' + r = p \text{ and } \forall s, t \in P \; r \ne qs + t.$$

By this relation we expect that whenever a process *q* acts on a process *p*, there exists a unique process *p*' which is related to *p*, via *q*. Thus, given a process $q \in P$, the transition relation $\xrightarrow{q}$ between the processes is deterministic.

This transition relation induces a transformation semigroup over *P* as shown in the following: For $q \in P$, define the partial function $f_q : P \to P$ by $f_q(p) = p' \Leftrightarrow p \xrightarrow{q} p'$, for $p \in P$. Since the transition relation $\xrightarrow{q}$ is deterministic, for each $q \in P$, f_q is well-defined.

Theorem 3.1. *The basic process algebra P, with respect to trace semantics, can be embedded in* $\mathcal{M}(P)$, *the monoid of all partial functions on P.*

Proof. Define a map $F: P \to \mathcal{M}(P)$ by $F(q) = f_q$, where $f_q : P \to P$ is the partial function given by

$$f_q(p) = p' \Leftrightarrow p \xrightarrow{q} p'.$$

Now, it is routine to verify that F is a one-one homomorphism so that P can be embedded in $\mathcal{M}(P)$. □

Remark 3.2. If P is a BPA with trace semantics, then (P, P) is a transformation semigroup.

To study the skeleton space $\mathfrak{I}$ of the transformation semigroup of P, let us observe $\mathfrak{I}^0$, the space of all s-images of P. That is, $\mathfrak{I}^0 = \{Ps \mid s \in P\} = \{Pf_q \mid q \in P\}$, from Theorem 3.1. Here for $q \in P$,

$$Pf_q = \{f_q(p) \mid p \in P\} = \{p' \mid p \xrightarrow{q} p', \text{ for some } p \in P\}.$$

Since for each $p' \in P$, qp' is also in P so that $qp' \xrightarrow{q} p'$, we have $Pf_q = P$, for all $q \in P$. Hence $\mathfrak{I}^0 = \{P\}$, a singleton. Consequently, the skeleton space of P is

$$\mathfrak{I} = \{P\} \cup \bigcup_{p \in P} \{\{p\}\}.$$

Further, note that if a process p' is an yield of a process p via an action of a nonempty process q, then p cannot be an yield of p' under the action of any process of P. Therefore, for each $p \in P$, $\{p\}$ is an equivalence class in $\mathfrak{I}$ with respect to the equivalence relation $\sim$. As a result, $\{P\}$ is an equivalence class of $\mathfrak{I}$. Thus the skeleton space of P consists of the whole set P and all singletons of P. Moreover, any two distinct members of $\mathfrak{I}$ will be in distinct equivalence classes.

In order to estimate the height of the transformation semigroup of P, let us first recall the following definition:

Definition 3.3. An integer valued function $h : \mathfrak{I} \to \mathbb{Z}$ is said to be a *height function* on a transformation semigroup (P, S) if it satisfies the following:

1. For all singletons $\{p\} \in \mathfrak{I}, h(\{p\}) = 0$;
2. For all $A, B \in \mathfrak{I}, A \sim B \Rightarrow h(A) = h(B)$;
3. For all $A, B \in \mathfrak{I}, A < B \Rightarrow h(A) < h(B)$;
4. If $0 \le i \le h(P)$, then there exists $A \in \mathfrak{I}$ such that $h(A) = i$.

Here, by $A < B$ we mean $A \le B$ but $B \le A$ does not hold. Given a height function h on (P, S), $h(P)$ is called as *height* of (P, S).

From the Definition 3.3, one may observe that the transformation semigroup of the basic process algebra P, with respect to trace semantics, is of height one, in which all singletons of height zero and then the whole set P is of height one. Thus the skeleton space of P is very simple.

4. CONCLUSION

Process algebras of ACP-style are equational reformulations of CCS which study the behavior of concurrent systems of interacting processes. The natural transition relation between the processes of process algebra induces a transition system for process algebras. In this work, we formulate a transformation semigroup of the transition system for basic process algebra with respect to trace semantics and investigate its skeleton. We observe that the skeleton is very simple to understand. In fact, we notice that the equivalence classes in the skeleton are precisely singletons of the process algebra and the whole process algebra. Thus, the height of the transformation semigroup is one – in which all singletons are of height zero and the whole set of height one. This work completes two important steps in studying the holonomy decomposition of process algebras. To give the complete holonomy decomposition of process algebras, one has to pursue the remaining step, viz. determining the holonomy groups of the respective equivalence classes of the skeleton.

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