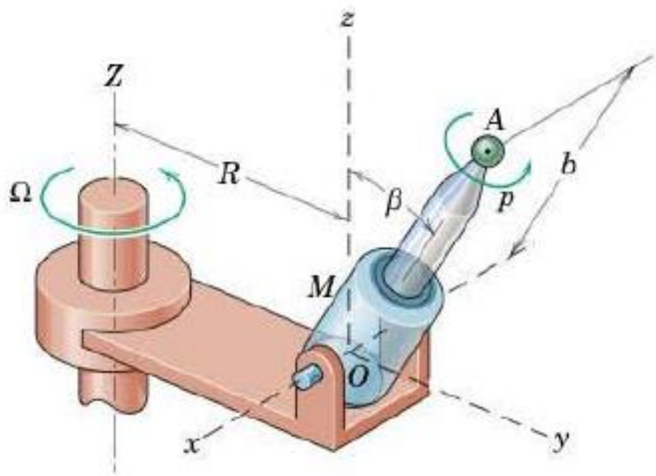
Ques.1 – The small motor *M* is pivoted about the *x*-axis through *O* and gives its shaft *OA* a constant speed *p* rad/s in the direction shown relative to its housing. The entire unit is then set into rotation about the vertical *Z*-axis at the constant angular velocity Ω rad/s. simultaneously, the motor pivots about the x-axis at the constant rate $\dot{\beta}$ for an interval of motion. Determine the angular acceleration α of the shaft *OA* in term of β. Express your result in term of the unit vector for the rotating *x*-*y*-*z* axes.



SOL. 1 Angular velocity of OA is $\underline{\omega} = -\dot{\beta}\underline{i} + p\sin\beta\underline{j} + (p\cos\beta + \Omega)\underline{k}$

Eq. 7/7a, $[\ ] = \underline{\omega}$, $\left(\frac{d[\ ]}{dt}\right)_{XYZ} = \left(\frac{d[\ ]}{dt}\right)_{xyz} + \underline{\Omega} \times [\ ]$

$$\left(\frac{d\omega}{dt}\right)_{xyz} = \underline{0} + p\dot{\beta}\cos\beta\,\underline{j} + (-p\dot{\beta}\sin\beta + 0)\underline{k}$$

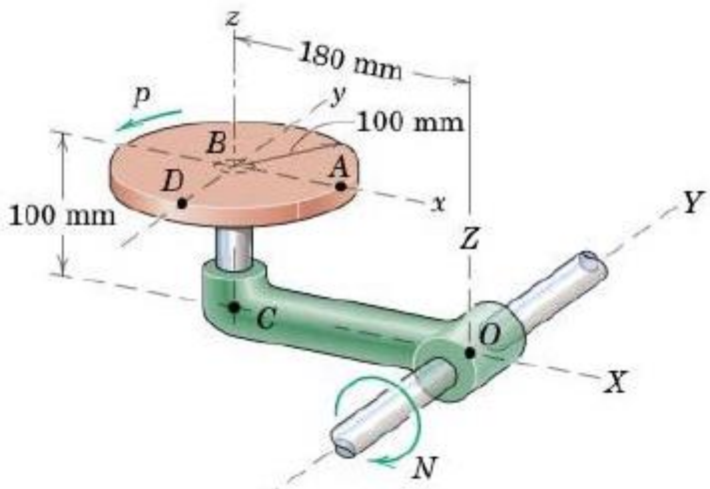
$$\underline{\Omega} \times \underline{\omega} = \Omega\underline{k} \times (-\dot{\beta}\underline{i} + p\sin\beta\underline{j} + [p\cos\beta + \Omega]\underline{k})$$

$$= -\Omega\dot{\beta}\underline{j} - \Omega p\sin\beta\,\underline{i}$$

so $\underline{\alpha} = (p\dot{\beta}\cos\beta - \Omega\dot{\beta})\underline{j} - \Omega p\sin\beta\,\underline{i} - p\dot{\beta}\sin\beta\,\underline{k}$

$$\underline{\alpha} = -\Omega p\sin\beta\,\underline{i} + \dot{\beta}(p\cos\beta - \Omega)\underline{j} - p\dot{\beta}\sin\beta\,\underline{k}$$

Ques.2 – The circular disk of 100-mm radius rotates about its *z*-axis at the constant speed $p = 240$ rev/min, and arm *OCB* rotates about *Y*-axis at the constant speed $N = 30$rev/min. Determine the velocity **v** and acceleration **a** of point *A* on the disk as it passes the position shown. Use reference axes *x-y-z* attached to arm *OCB.*



2. SOL

$\mathbf{\Omega}$ = angular velocity of axes x-y-z = $\frac{2\pi N}{60}\mathbf{j} = \pi\mathbf{j}\ \frac{rad}{s}$

$\mathbf{v} = \mathbf{v}_A = \mathbf{v}_B + \mathbf{\Omega}\times\mathbf{r}_{A/B} + \mathbf{v}_{rel}$

where $\mathbf{v}_B = \pi\mathbf{j}\times\mathbf{r}_{OB} = \pi\mathbf{j}\times(-0.18\mathbf{i} + 0.1\mathbf{k}) = \pi(0.1\mathbf{i} + 0.18\mathbf{k})$ m/s

$\mathbf{\Omega}\times\mathbf{r}_{A/B} = \pi\mathbf{j}\times 0.1\mathbf{i} = -0.1\pi\mathbf{k}$ m/s

$\mathbf{v}_{rel} = p\mathbf{k}\times\mathbf{r}_{A/B} = \frac{240(2\pi)}{60}\mathbf{k}\times 0.1\mathbf{i} = 0.8\pi\mathbf{j}$ m/s

Collect terms & get $\underline{\mathbf{v} = \pi(0.1\mathbf{i} + 0.8\mathbf{j} + 0.08\mathbf{k})\ \text{m/s}}$

$\mathbf{a} = \mathbf{a}_A = \mathbf{a}_B + \dot{\mathbf{\Omega}}\times\mathbf{r}_{A/B} + \mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r}_{A/B}) + 2\mathbf{\Omega}\times\mathbf{v}_{rel} + \mathbf{a}_{rel}$; $\dot{\mathbf{\Omega}} = \mathbf{0}$

where $\mathbf{a}_B = \mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r}_{B/O}) = \pi\mathbf{j}\times(\pi\mathbf{j}\times[-0.18\mathbf{i} + 0.1\mathbf{k}])$

$= \pi^2(0.18\mathbf{i} - 0.1\mathbf{k})$ m/s²

$\mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r}_{A/B}) = \pi\mathbf{j}\times(-0.1\pi\mathbf{k}) = -0.1\pi^2\mathbf{i}$ m/s²

$2\mathbf{\Omega}\times\mathbf{v}_{rel} = 2\pi\mathbf{j}\times 0.8\pi\mathbf{j} = \mathbf{0}$

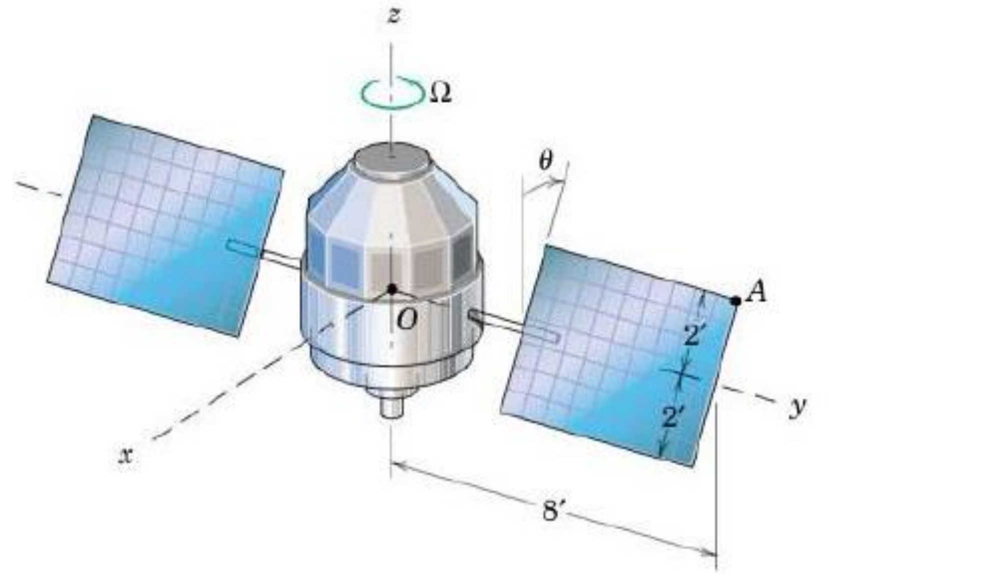
$\mathbf{a}_{rel} = \dot{p}\mathbf{k}\times\mathbf{r}_{A/B} + r_{A/B}\,p^2(-\mathbf{i}) = \mathbf{0} - (8\pi)^2 0.1\mathbf{i} = -6.4\pi^2\mathbf{i}\ \frac{m}{s^2}$

Collect terms & get

$\mathbf{a} = -0.1\pi^2\mathbf{i} - 6.4\pi^2\mathbf{i} + 0.18\pi^2\mathbf{i} - 0.1\pi^2\mathbf{k}$

$\underline{\mathbf{a} = -\pi^2(6.32\mathbf{i} + 0.1\mathbf{k})\ \text{m/s}^2}$

Ques.3- The center O of the spacecraft is moving through space with a constant velocity. During the period of motion prior to stabilization, the spacecraft has a constant rotational rate Ω = ½ rad/sec about its *z*-axis. The x-y-z axes are attached to the body of the craft, and the solar panels rotate about the y-axis at the constant rate $\dot{\theta}$ = ¼ rad/sec with respect to the spacecraft. If ω is the absolute angular velocity of the solar panels, determine $\dot{\omega}$. Also find the acceleration of point A when θ = 30º.



Sol. 3

Angular velocity of axes $\underline{\Omega} = \Omega \underline{k}$

" " " panels $\underline{\omega} = -\dot{\theta}\underline{j} + \Omega \underline{k}$

$\dot{\underline{\omega}} = -\dot{\theta}\dot{\underline{j}} + \Omega \dot{\underline{k}} = -\dot{\theta}(\underline{\Omega} \times \underline{j}) + \Omega(\underline{\Omega} \times \underline{k}) = \underline{\Omega} \times \underline{\omega} = \Omega\dot{\theta}\underline{i}$

$= \frac{1}{2}\frac{1}{4}\underline{i} = \underline{\frac{1}{8}\underline{i} \text{ rad/sec}^2}$

$\underline{a}_A = \underline{a}_o + \dot{\underline{\Omega}} \times \underline{r}_{A/o} + \underline{\Omega} \times (\underline{\Omega} \times \underline{r}_{A/o}) + 2\underline{\Omega} \times \underline{v}_{rel} + \underline{a}_{rel}$

$\underline{a}_o = \underline{0}$; $\quad \underline{\Omega} \times \underline{r}_{A/o} = \frac{1}{2}\underline{k} \times (-\underline{i} + 8\underline{j} + \sqrt{3}\underline{k}) = -\frac{1}{2}\underline{j} - 4\underline{i} \ \frac{ft}{sec}$

$\underline{\Omega} \times (\underline{\Omega} \times \underline{r}_{A/o}) = \frac{1}{2}\underline{k} \times (-\frac{1}{2}\underline{j} - 4\underline{i}) = \frac{1}{4}\underline{i} - 2\underline{j}$ ft/sec^2

$2\underline{\Omega} \times \underline{v}_{rel} = 2(\frac{1}{2}\underline{k}) \times (-\frac{\sqrt{3}}{4}\underline{i} - \frac{1}{4}\underline{k}) = -\frac{\sqrt{3}}{4}\underline{j}$ ft/sec^2

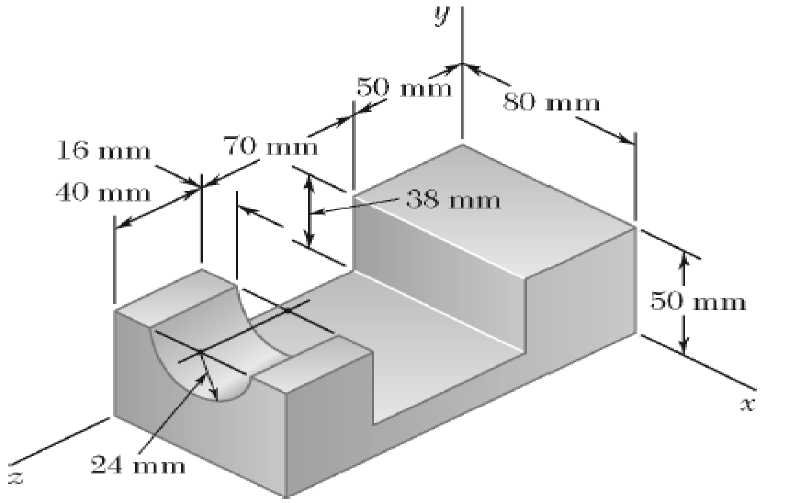
$\underline{a}_{rel} = 2(\frac{1}{4})^2(\frac{1}{2}\underline{i} - \frac{\sqrt{3}}{2}\underline{k}) = \frac{1}{16}\underline{i} - \frac{\sqrt{3}}{16}\underline{k}$ ft/sec^2

$\underline{a}_A = (\frac{1}{4} + \frac{1}{16})\underline{i} + (-2 - \frac{\sqrt{3}}{4})\underline{j} - \frac{\sqrt{3}}{16}\underline{k}$

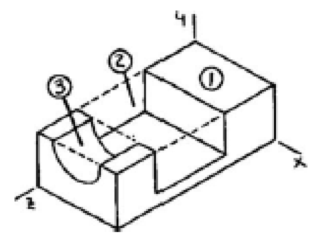
$= \underline{0.313\underline{i} - 2.43\underline{j} - 0.1083\underline{k} \text{ ft/sec}^2}$

with $a_A = 2.45$ ft/sec^2

Ques.4– Determine the mass moment of inertia of the steel fixture shown with respect to *(a)* the *x*-axis, (b) the y-axis, (c) the z-axis. (The density of steel is 7850 kg/m^3.)



SOLUTION



First compute the mass of each component. We have

$$m = \rho_{ST} V$$

Then
$$m_1 = 7850 \text{ kg/m}^3 \times (0.08 \times 0.05 \times 0.160) \text{ m}^3$$
$$= 5.02400 \text{ kg}$$

$$m_2 = 7850 \text{ kg/m}^3 \times (0.08 \times 0.038 \times 0.07) \text{ m}^3 = 1.67048 \text{ kg}$$

$$m_3 = 7850 \text{ kg/m}^3 \times \left(\frac{\pi}{2} \times 0.024^2 \times 0.04\right) \text{m}^3 = 0.28410 \text{ kg}$$

Using Figure 9.28 for components 1 and 2 and the equations derived above for component 3, we have

(*a*) $I_x = (I_x)_1 - (I_x)_2 - (I_x)_3$

$$= \left\{\frac{1}{12}(5.02400 \text{ kg})[(0.05)^2 + (0.16)^2] \text{ m}^2 + (5.02400 \text{ kg})\left[\left(\frac{0.05}{2}\right)^2 + \left(\frac{0.16}{2}\right)^2\right] \text{m}^2\right\}$$

$$- \left\{\frac{1}{12}(1.67048 \text{ kg})[(0.038)^2 + (0.07)^2] \text{ m}^2 + (1.67048 \text{ kg})\left[\left(0.05 - \frac{0.038}{2}\right)^2 + \left(0.05 + \frac{0.07}{2}\right)^2\right] \text{m}^2\right\}$$

$$- \left\{(0.28410 \text{ kg})\left[\left(\frac{1}{4} - \frac{16}{9\pi^2}\right)(0.024)^2 + \frac{1}{12}(0.04)^2\right] \text{m}^2\right.$$

$$\left. + (0.28410 \text{ kg})\left[\left(0.05 - \frac{4 \times 0.024}{3\pi}\right)^2 + \left(0.16 - \frac{0.04}{2}\right)^2\right] \text{m}^2\right\}$$

$$= [(11.7645 + 35.2936) - (0.8831 + 13.6745) - (0.0493 + 6.0187)] \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

$$= (47.0581 - 14.5576 - 6.0680) \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

$$= 26.4325 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

or $I_x = 26.4 \times 10^{-3} \text{ kg} \cdot \text{m}^2$ ◀

(*b*) $I_y = (I_y)_1 - (I_y)_2 - (I_y)_3$

$$= \left\{\frac{1}{12}(5.02400\text{ kg})[(0.08)^2 + (0.16)^2]\text{ m}^2 + (5.02400\text{ kg})\left[\left(\frac{0.08}{2}\right)^2 + \left(\frac{0.16}{2}\right)^2\right]\text{m}^2\right\}$$

$$-\left\{\frac{1}{12}(1.67048\text{ kg})[(0.08)^2 + (0.07)^2]\text{ m}^2 + (1.67048\text{ kg})\left[\left(\frac{0.08}{2}\right)^2 + \left(0.05 + \frac{0.07}{2}\right)^2\right]\text{m}^2\right\}$$

$$-\left\{\frac{1}{12}(0.28410\text{ kg})[3(0.024)^2 + (0.04)^2]\text{ m}^2 + (0.28410\text{ kg})\left[\left(\frac{0.08}{2}\right)^2 + \left(0.16 - \frac{0.04}{2}\right)^2\right]\text{m}^2\right\}$$

$$= [(13.3973 + 40.1920) - (1.5730 + 14.7420) - (0.0788 + 6.0229)] \times 10^{-3}\text{ kg}\cdot\text{m}^2$$

$$= (53.5893 - 16.3150 - 6.1017) \times 10^{-3}\text{ kg}\cdot\text{m}^2$$

$$= 31.1726 \times 10^{-3}\text{ kg}\cdot\text{m}^2$$

or $I_y = 31.2 \times 10^{-3}\text{ kg}\cdot\text{m}^2$ ◀

(*c*) $I_z = (I_z)_1 - (I_z)_2 - (I_z)_3$

$$= \left\{\frac{1}{12}(5.02400\text{ kg})[(0.08)^2 + (0.05)^2]\text{ m}^2 + (5.02400\text{ kg})\left[\left(\frac{0.08}{2}\right)^2 + \left(\frac{0.05}{2}\right)^2\right]\text{m}^2\right\}$$

$$-\left\{\frac{1}{12}(1.67048\text{ kg})[(0.08)^2 + (0.038)^2]\text{ m}^2 + (1.67048\text{ kg})\left[\left(\frac{0.08}{2}\right)^2 + \left(0.05 - \frac{0.038}{2}\right)^2\right]\text{m}^2\right\}$$

$$-\left\{(0.28410\text{ kg})\left(\frac{1}{2} - \frac{16}{9\pi^2}\right)(0.024\text{ m})^2 + (0.28410\text{ kg})\left[\left(\frac{0.08}{2}\right)^2 + \left(0.05 - \frac{4\times 0.024}{3\pi}\right)^2\right]\text{m}^2\right\}$$

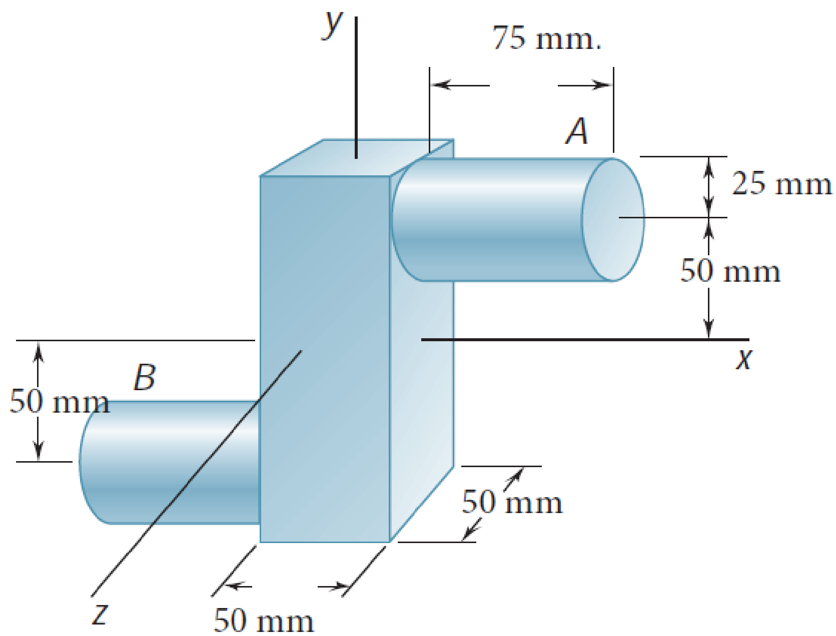
$$= [(3.7261 + 11.1784) - (1.0919 + 4.2781) - (0.0523 + 0.9049)] \times 10^{-3}\text{ kg}\cdot\text{m}^2$$

$$= (14.9045 - 5.3700 - 0.9572) \times 10^{-3}\text{ kg}\cdot\text{m}^2$$

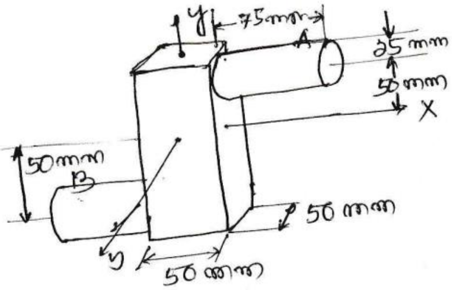
$$= 8.5773 \times 10^{-3}\text{ kg}\cdot\text{m}^2$$

or $I_z = 8.58 \times 10^{-3}\text{ kg}\cdot\text{m}^2$ ◀

Ques.5 – A steel forging consists of a 150 x 50 x 50-mm rectangular prism and two cylinders of diameter 50 mm and length 75 mm as shown. Determine the moments of inertia of the forging with respect to the coordinate axes, knowing that the specific weight of steel is 7850 kg/m^3.



8. A steel forging consists of a 150 X 50 X 50 mm rectangular prism and two cylinders of diameter 50 mm and Length 75 mm as shown. Determine the moments of inertia of the forging with respect to the coordinate axes, knowing that the density of steel is 7850 kg/m³



Solution :

Computation of masses

Prism $V = (0{\cdot}05\ m)(0{\cdot}05\ m)(0{\cdot}15 m) = 3{\cdot}75 \times 10^{-4} m^3$

$m = (7850\ kg/m^3)(3{\cdot}75 \times 10^{-4} m^3) = 2{\cdot}94\ kg$

Each cylinder

$V = \pi (0 \cdot 025\ m)^2 (0 \cdot 075\ m) = 1{\cdot}473 \times 10^{-4}\ m^3$

$m = (7850\ kg/m^3)(1{\cdot}473 \times 10^{-4}\ m^3) = 1{\cdot}16\ kg$

Moments of inertia: The moments of interetia of each component are computed ~~from~~ using the parallel axes theorem when necessary. Note that all lengths should be expressed meter.

Prism: $I_x = I_z = \frac{1}{12}(2{\cdot}94\ kg)\left[(0{\cdot}15\ m)^2 + (0{\cdot}05\ m)^2\right] = 6{\cdot}125 \times 10^{-3}\ kg \cdot m^2$

$I_y = \frac{1}{12}(2{\cdot}94\ kg)\left[(0{\cdot}05 m)^2 + (0{\cdot}05\ m)^2\right] = 1{\cdot}225 \times 10^{-3}\ kg \cdot m^2$

Each cylinder:

$I_x = \frac{1}{2} m a^2 + m\bar{y}^2 = \frac{1}{2}(1{\cdot}16\ kg)(0{\cdot}025\ m)^2 + (1{\cdot}16\ kg)(0{\cdot}05\ m)^2$

$= 3{\cdot}263 \times 10^{-3}\ kg \cdot m^2$

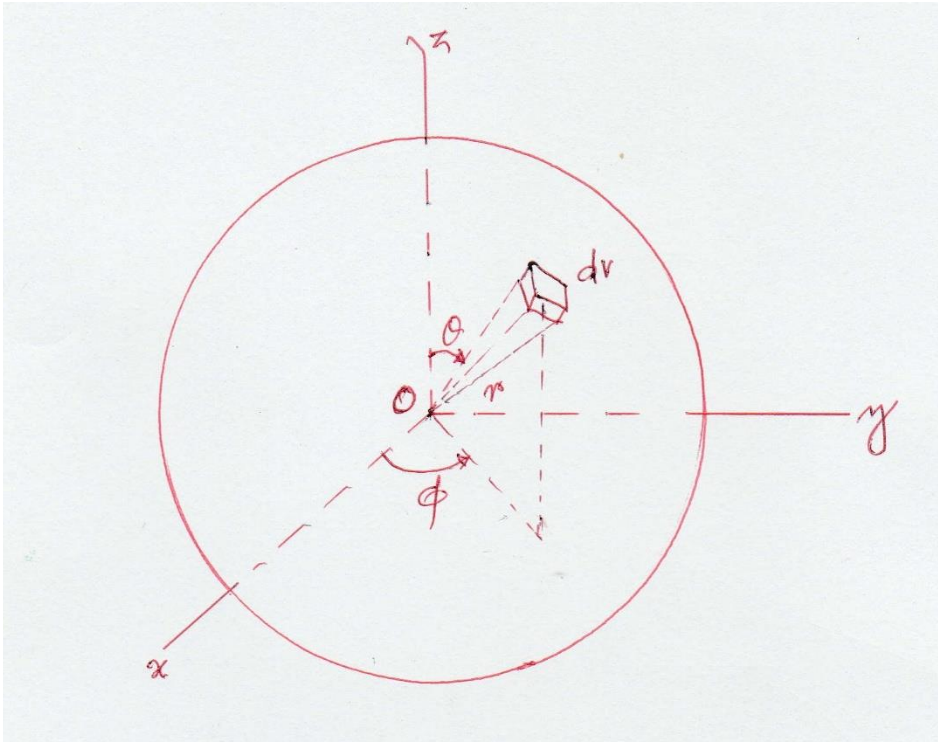
$I_y = \frac{1}{12} m(3a^2 + L^2) + m\bar{x}^2 = \frac{1}{12}(1{\cdot}16 kg)\left[3(0{\cdot}025\ m)^2 + (0{\cdot}075\ m)^2\right] + (1{\cdot}16 kg)(0 \cdot 0625\ m)^2$

$= 5{\cdot}256 \times 10^{-3}\ kg \cdot m^2$

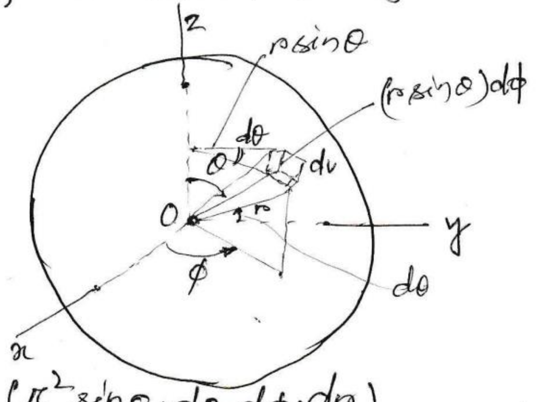
$I_z = \frac{1}{12} m(3a^2 + L^2) + m(\bar{x}^2 + \bar{y}^2) = \frac{1}{12}(1{\cdot}16\ kg)\left[3(0{\cdot}025 m)^2 + (0{\cdot}075 m)^2\right] + (1{\cdot}16 kg)\left[(0{\cdot}0625\ m)^2 + (0{\cdot}05 m)^2\right]$

$= 8{\cdot}156 \times 10^{-3}\ kg \cdot m^2$

Ques.6 Compute the component of inertia tensor at the center of a solid sphere of uniform density ρ as shown in Fig.



Q. Compute the components of the inertia tensor at the centre of a solid sphere of uniform density ρ as shown in fig.



Sol^n We shall first compute I_{yy}.
Using spherical coordinate, we have

$$I_{yy} = \iiint_V (x^2 + z^2)\, \rho\, dv$$

$$= \int_0^R \int_0^{2\pi} \int_0^{\pi} \left[(r \sin\theta \cdot \cos\phi)^2 + (r \cos\theta)^2\right] \rho \,(r^2 \sin\theta \cdot d\theta \cdot d\phi \cdot dr)$$

$$= \int_0^R \int_0^{2\pi} \int_0^{\pi} (r^4 \sin^3\theta \cdot \cos^2\phi)\, \rho\, d\theta\, d\phi\, dr + \int_0^R \int_0^{2\pi} \int_0^{\pi} (r^4 \cos^2\theta \cdot \sin\theta)\, \rho\, d\theta\, d\phi\, dr$$

$$= \rho \int_0^R \int_0^{2\pi} (r^4 \cos^2\phi) \left(\int_0^{\pi} \sin^3\theta\, d\theta\right) d\phi \cdot dr$$
$$+ \rho \int_0^R \int_0^{2\pi} r^4 \left(\int_0^{\pi} \cos^2\theta \cdot \sin\theta\, d\theta\right) d\phi\, dr$$

* dv is the volume of the small element

$$dv = (r \sin\theta\, d\phi)(dr)(r\, d\theta) = r^2 \sin\theta\, d\theta \cdot d\phi \cdot dr.$$

With the aid of integration formulas, we have

$$I_{yy} = \rho \int_0^R \int_0^{2\pi} r^4 \cos^2\phi \left[-\frac{1}{3}\cos\theta(\sin^2\theta + 2)\right]\Big|_0^{\pi} d\phi\, dr$$

$$+ \rho \int_0^R \int_0^{2\pi} r^4 \left(-\frac{\cos^3\theta}{3}\right)\Big|_0^{\pi} d\phi\, dr$$

$$= \rho \int_0^R \int_0^{2\pi} r^4 \cos^2\phi \cdot \frac{4}{3}\, d\phi\, dr + \rho \int_0^R \int_0^{2\pi} (r^4)\, \frac{2}{3}\, d\phi\, dr$$

Integrating next with respect to ϕ, we get

$$I_{yy} = \rho \int_0^R (r^4)\left(\frac{4}{3}\right)(\pi)\, dr + \rho \int_0^R (r^4)\left(\frac{2}{3}\right)(2\pi)\, dr$$

Finally we get

$$I_{yy} = \rho \frac{R^5}{5} \frac{4}{3}\pi + \rho \frac{R^5}{5} \frac{4}{3}\pi$$

$$\therefore \quad I_{yy} = \frac{8}{15}\rho\pi R^5$$

But $M = \rho \frac{4}{3}\pi R^3$, Hence $I_{yy} = \frac{2}{5} MR^2$

Because of the point symmetry about point O, we can also say that

$$I_{xx} = I_{zz} = \frac{2}{5} MR^2$$

Because the coordinate planes are all planes of symmetry for the mass distribution, the product of inertia are zero. Thus the inertia tensor can be given as

$$I_{ij} = \begin{bmatrix} \frac{2}{5}MR^2 & 0 & 0 \\ 0 & \frac{2}{5}MR^2 & 0 \\ 0 & 0 & \frac{2}{5}MR^2 \end{bmatrix}$$