Kinetics of Plane Motion of Rigid bodies

motion of Rigid body$|_t$ = translational motion + Rotational motion

Translational motion → actual instanteous velocity at any point of the body.

Angular velocity → (ω) ∋ Its axis of rotation through the chosen point.

→ Center of Mass of the rigid body.

Total external force act on the center of mass

$$\boxed{F = m\dot{V}_c}$$

m = Total mass of the rigid body.

~~Total~~ Due to angular velocity

$$\boxed{M_A = \dot{H}_A}$$ (Rate of change of angular momentum)

$$H = I\omega$$

$$H = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

$$\boxed{M_A = \dot{H}_A}$$ Validity

(1) The mass center

(2) Points fixed or moving with constant V at time t in inertial space

(points having zero acceleration at time t relative to inertial reference XYZ)

(3) A point accelerating toward or away from the mass center.

Basic Equation

$$\left(\frac{dA}{dt}\right)_{XYZ} = \left(\frac{dA}{dt}\right)_{xyz} + \omega \times A$$

Where ω is the angular velocity of xyz relative to XYZ

Now Ω angular velocity of xyz w.r.t. XYZ

$$\left(\frac{dH_A}{dt}\right)_{XYZ} = \left(\frac{dH_A}{dt}\right)_{xyz} + \Omega \times H_A$$

$$\frac{d}{dt}(H_A)_{xy} = \frac{d}{dt}\left[\sout{\omega_z I_{zx}} + I\omega \right]$$

$$= \sout{\frac{d\,\omega_z}{dt}} \quad I \frac{d}{dt}(\dot{\omega})$$

$$\Omega \times H_A = \Omega \times [I]\{\omega\}$$

$$= (\Omega_x i + \Omega_y j + \Omega_z \hat{k}) \times \Big[\underbrace{(I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z)}_{H_x} i$$

$$+ \underbrace{(-I_{xy}\omega_x + I_{yy}\omega_y - I_{yz}\omega_z)}_{H_y} j +$$

$$\underbrace{(-I_{zx}\omega_x - I_{zy}\omega_y + I_{zz}\omega_z)}_{H_z} \hat{k} \Big) \Big]$$

$$= \begin{vmatrix} i & j & k \\ \Omega_x & \Omega_y & \Omega_z \\ H_x & H_y & H_z \end{vmatrix}$$

$$= i(\Omega_y H_z - \Omega_z H_y) + j[H_x \Omega_z - \Omega_x H_z) + k(\Omega_x H_y - H_x \Omega_y)$$

Now

$$M_A = (I_{xx}\dot{\omega}_x - I_{xy}\dot{\omega}_y - I_{xz}\dot{\omega}_z) i + [(-\Omega_y \omega_x I_{zx} - \Omega_y \omega_y I_{zy} + \Omega_y \omega_z I_{zz} - [\Omega_z(-I_{xy})\omega_x + \Omega_z \omega_y I_{yy} - \Omega_z \omega_z I_{yz})] i + [(-I_{yx}\dot{\omega}_x + I_{yy}\dot{\omega}_y - I_{yz}\dot{\omega}_z) j$$

$$+ (\Omega_z \omega_x I_{xx} - \Omega_z \omega_y I_{xy} - \Omega_z \omega_z I_{xz} - (-\Omega_x I_{zx}\omega_x - \Omega_x \omega_y I_{zy} + \Omega_x \omega_z I_{zz}) j] + [-I_{zx}\dot{\omega}_x - I_{zy}\dot{\omega}_y + I_{zz}\dot{\omega}_z + \Omega_x \omega_x(-I_{yx}) + \Omega_x \omega_y I_{yy} - \Omega_x \omega_z I_{yz} - [-\Omega_y \omega_x I_{xx} - \Omega_y \omega_y I_{xy} - \Omega_y \omega_z I_{xz}] \hat{k}$$

$(M_x \hat{i} + M_y \hat{j} + M_z \hat{k}) =$

$$\left[I_{xx}\dot{\omega}_x + \Omega_y \omega_z (I_{zz} - I_{yy}) + I_{xy}(\Omega_z \omega_x - \dot{\omega}_y) - I_{xz}(\dot{\omega}_z + \Omega_y \omega_x) - I_{yz}(\Omega_y \omega_y - \Omega_z \omega_z)\right]\hat{i}$$

$$+ \left[I_{yy}\dot{\omega}_y + \Omega_x \omega_x (I_{xx} - I_{zz}) + I_{yz}(\Omega_x \omega_y - \dot{\omega}_z) - I_{yx}(\dot{\omega}_x + \Omega_z \omega_y) - I_{zx}(\Omega_z \omega_z - \Omega_x \omega_x)\right]\hat{j}$$

$$+ \left[I_{zz}\dot{\omega}_z + \Omega_x \omega_y (I_{yy} - I_{xx}) + I_{zx}(\Omega_y \omega_z - \dot{\omega}_x) - I_{zy}(\Omega_x \omega_z + \dot{\omega}_y) - I_{xy}(\Omega_x \omega_x - \Omega_y \omega_y)\right]\hat{k}$$

For plane body:

$$\Omega_x = \Omega_y = 0$$
$$\omega_x = \omega_y = 0$$

$$M_x \hat{i} + M_y \hat{j} + M_z \hat{k} = (-I_{xz}\dot{\omega}_z + I_{yz}\Omega_z \omega_z)\hat{i} + (-I_{yz}\dot{\omega}_z - I_{zx}\Omega_z \omega_z)\hat{j} + (I_{zz}\dot{\omega}_z)\hat{k}$$

For Principal axes

$$(M_x \hat{i} + M_y \hat{j} + M_z \hat{k}) = (I_{xx}\dot{\omega}_x + \Omega_y \omega_z (I_{zz} - I_{yy}))\hat{i} + (I_{yy}\dot{\omega}_y + \Omega_x \omega_x (I_{xx} - I_{zz}))\hat{j} + (I_{zz}\dot{\omega}_z + \Omega_x \omega_y (I_{yy} - I_{xx}))\hat{k}$$

~~when $\Omega_z = \omega_z$~~

Angular velocity as given by ω ($\dot{\theta}$) is always taken relative to the inertial reference XYZ, whereas the moment of forces ($(M_A)_x$, $(M_A)_y$ & $(M_A)_z$) as well as the inertia tensor components are always taken about the xyz fixed to the body A.

$$T = I\alpha$$
$$T = I\ddot{\theta}$$
$$\Rightarrow (M_A)_z = I_{zz}\dot{\omega}$$

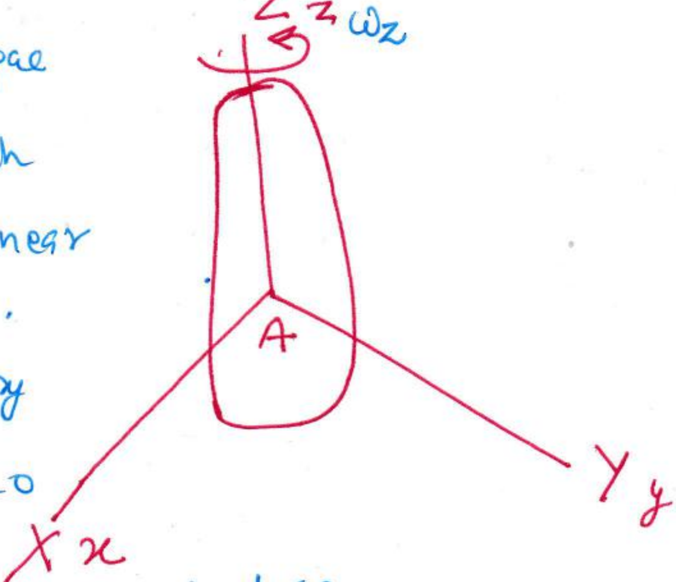
Pure Rotation of a Body of Revolution about its axis of Revolution.

XYZ – fixed in inertial space

xyz – fixed to body such that z axis is collinear with axis of revolution.

Plane zy is a plane of symmetry

$I_{xy} = I_{xz} = 0$, $I_{yz} = 0$



Uniform Rigid body of revolution time t

$M_z = I_{zz}\dot{\omega}_z$

$M_x = 0$

$M_y = 0$

Eighth Edition

VECTOR MECHANICS FOR ENGINEERS:

DYNAMICS

CHAPTER

18

Ferdinand P. Beer

E. Russell Johnston, Jr.

Lecture Notes:

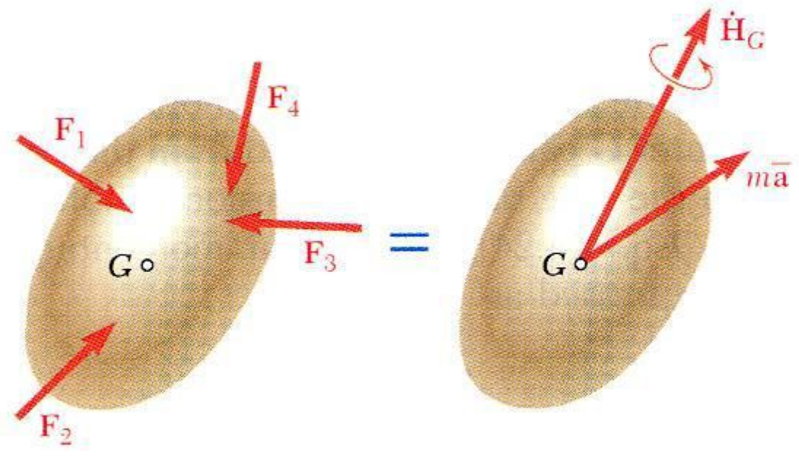
J. Walt Oler

Texas Tech University

Kinematics of Rigid Bodies in Three Dimensions

Contents

Introduction

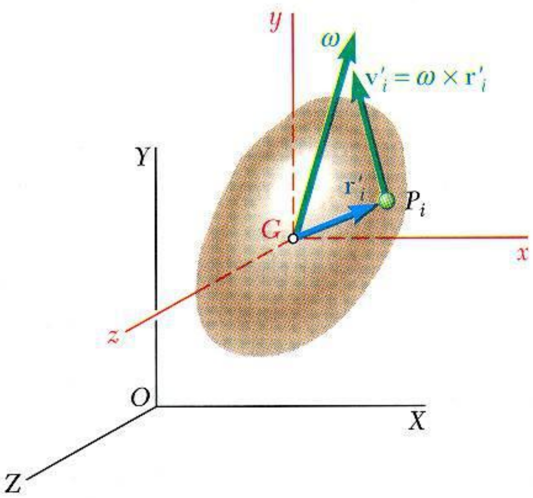


$$\sum \vec{F} = m\vec{\bar{a}}$$

$$\sum \vec{M}_G = \dot{\vec{H}}_G$$

- The fundamental relations developed for the plane motion of rigid bodies may also be applied to the general motion of three dimensional bodies.
- The relation $\vec{H}_G = \bar{I}\vec{\omega}$ which was used to determine the angular momentum of a rigid slab is not valid for general three dimensional bodies and motion.
- The current chapter is concerned with evaluation of the angular momentum and its rate of change for three dimensional motion and application to effective forces, the impulse-momentum and the work-energy principles.

Rigid Body Angular Momentum in Three Dimensions



- Angular momentum of a body about its mass center,

$$\vec{H}_G = \sum_{i=1}^{n}\left(\vec{r}_i' \times \vec{v}_i \Delta m_i\right) = \sum_{i=1}^{n}\left[\vec{r}_i' \times \left(\vec{\omega} \times \vec{r}_i'\right)\Delta m_i\right]$$

- The x component of the angular momentum,

$$H_x = \sum_{i=1}^{n}\left[y_i\left(\vec{\omega} \times \vec{r}_i'\right)_z - z_i\left(\vec{\omega} \times \vec{r}_i'\right)_y\right]\Delta m_i$$

$$= \sum_{i=1}^{n}\left[y_i\left(\omega_x y_i - \omega_y x_i\right) - z_i\left(\omega_z x_i - \omega_x z_i\right)\right]\Delta m_i$$

$$= \omega_x \sum_{i=1}^{n}\left(y_i^2 + z_i^2\right)\Delta m_i - \omega_y \sum_{i=1}^{n} x_i y_i \Delta m_i - \omega_z \sum_{i=1}^{n} z_i x_i \Delta m_i$$

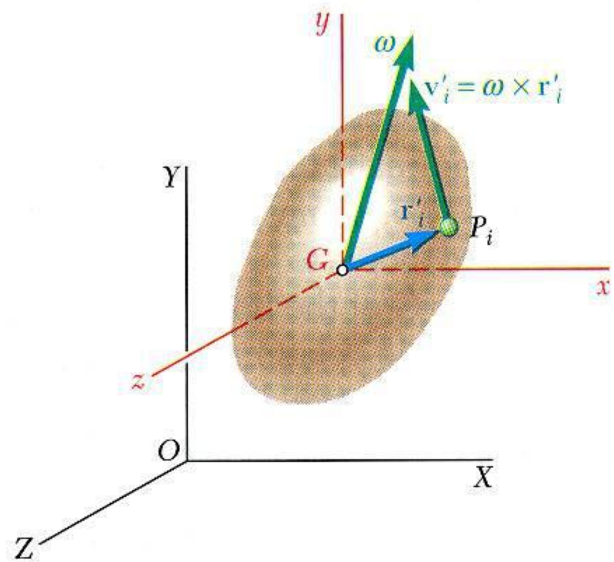
$$H_x = \omega_x \int\left(y^2 + z^2\right)dm - \omega_y \int xy\,dm - \omega_z \int zx\,dm$$

$$= +\bar{I}_x \omega_x - \bar{I}_{xy}\omega_y - \bar{I}_{xz}\omega_z$$

$$H_y = -\bar{I}_{yx}\omega_x + \bar{I}_y \omega_y - \bar{I}_{yz}\omega_z$$

$$H_z = -\bar{I}_{zx}\omega_x - \bar{I}_{zy}\omega_y + \bar{I}_z \omega_z$$

Rigid Body Angular Momentum in Three Dimensions



$$H_x = +\bar{I}_x \omega_x - \bar{I}_{xy}\omega_y - \bar{I}_{xz}\omega_z$$
$$H_y = -\bar{I}_{yx} \omega_x + \bar{I}_{y}\omega_y - \bar{I}_{yz}\omega_z$$
$$H_z = -\bar{I}_{zx} \omega_x - \bar{I}_{zy}\omega_y + \bar{I}_{z}\omega_z$$

- Transformation of $\vec{\omega}$ into $\vec{H}_G$ is characterized by the inertia tensor for the body,

$$\begin{pmatrix} +\bar{I}_x & -\bar{I}_{xy} & -\bar{I}_{xz} \\ -\bar{I}_{yx} & +\bar{I}_y & -\bar{I}_{yz} \\ -\bar{I}_{zx} & -\bar{I}_{zy} & +\bar{I}_z \end{pmatrix}$$

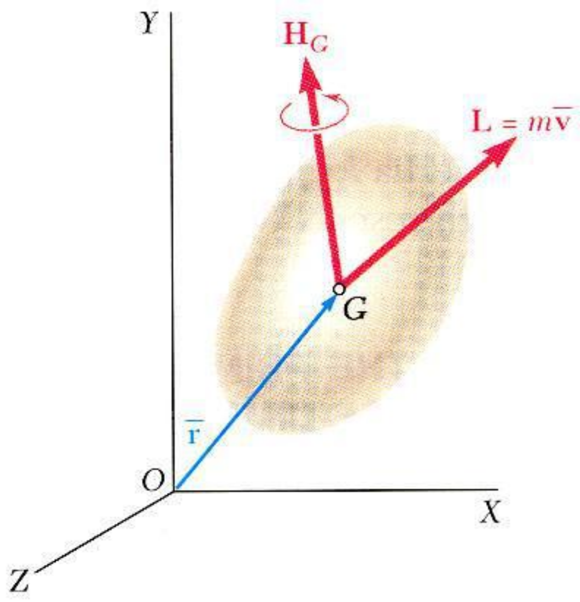
- With respect to the principal axes of inertia,

$$\begin{pmatrix} \bar{I}_{x'} & 0 & 0 \\ 0 & \bar{I}_{y'} & 0 \\ 0 & 0 & \bar{I}_{z'} \end{pmatrix}$$

$$H_{x'} = \bar{I}_{x'}\omega_{x'} \quad H_{y'} = \bar{I}_{y'}\omega_{y'} \quad H_{z'} = \bar{I}_{z'}\omega_{z'}$$

- The angular momentum $\vec{H}_G$ of a rigid body and its angular velocity $\vec{\omega}$ have the same direction if, and only if, $\vec{\omega}$ is directed along a principal axis of inertia.

Rigid Body Angular Momentum in Three Dimensions



- The momenta of the particles of a rigid body can be reduced to:

$$\vec{L} = \text{linear momentum}$$
$$= m\vec{\bar{v}}$$

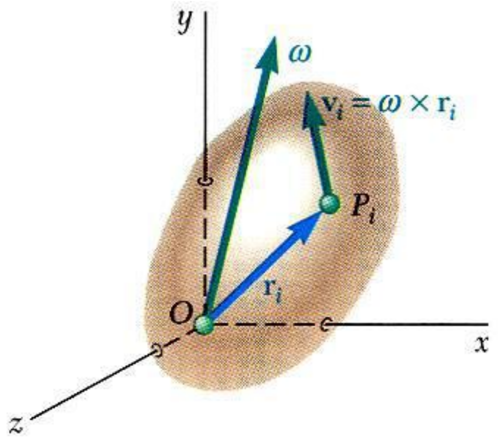
$$\vec{H}_G = \text{angular momentum about } G$$

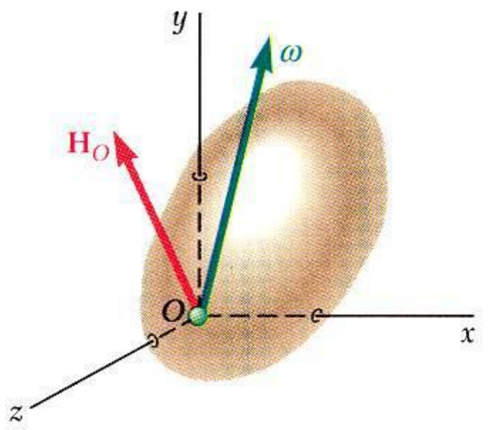
$$H_x = +\bar{I}_x\omega_x - \bar{I}_{xy}\omega_y - \bar{I}_{xz}\omega_z$$
$$H_y = -\bar{I}_{yx}\omega_x + \bar{I}_y\omega_y - \bar{I}_{yz}\omega_z$$
$$H_z = -\bar{I}_{zx}\omega_x - \bar{I}_{zy}\omega_y + \bar{I}_z\omega_z$$

- The angular momentum about any other given point *O* is

$$\vec{H}_O = \vec{\bar{r}} \times m\vec{\bar{v}} + \vec{H}_G$$

Rigid Body Angular Momentum in Three Dimensions





- The angular momentum of a body constrained to rotate about a fixed point may be calculated from

$$\vec{H}_O = \vec{r} \times m\vec{v} + \vec{H}_G$$

- Or, the angular momentum may be computed directly from the moments and products of inertia with respect to the *Oxyz* frame.

$$\vec{H}_O = \sum_{i=1}^{n} \left(\vec{r}_i \times \vec{v}_i \Delta m\right)$$

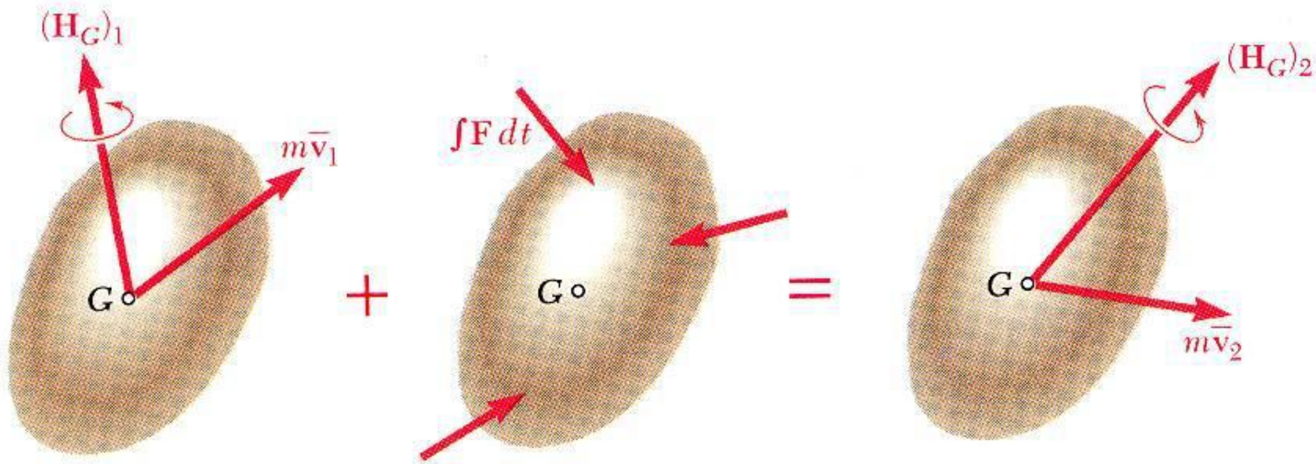
$$= \sum_{i=1}^{n} \left[\vec{r}_i \times \left(\vec{\omega} \times \vec{r}_i\right) \Delta m_i\right]$$

$$H_x = +I_x \omega_x - I_{xy}\omega_y - I_{xz}\omega_z$$

$$H_y = -I_{yx}\omega_x + I_y \omega_y - I_{yz}\omega_z$$

$$H_z = -I_{zx}\omega_x - I_{zy}\omega_y + I_z \omega_z$$

Principle of Impulse and Momentum

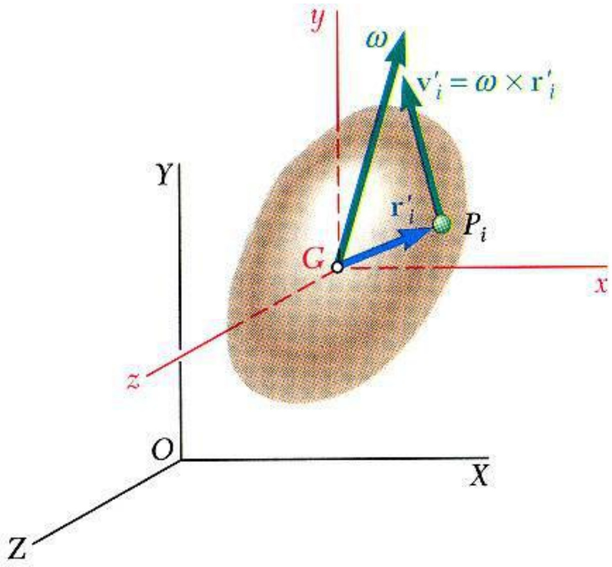


- The principle of impulse and momentum can be applied directly to the three-dimensional motion of a rigid body,

$$\textit{Syst Momenta}_1 + \textit{Syst Ext Imp}_{1\text{-}2} = \textit{Syst Momenta}_2$$

- The free-body diagram equation is used to develop component and moment equations.
- For bodies rotating about a fixed point, eliminate the impulse of the reactions at *O* by writing equation for moments of momenta and impulses about *O*.

Kinetic Energy



- Kinetic energy of particles forming rigid body,

$$T = \tfrac{1}{2} m\bar{v}^2 + \tfrac{1}{2}\sum_{i=1}^{n} \Delta m_i \bar{v}_i'^2$$

$$= \tfrac{1}{2} m\bar{v}^2 + \tfrac{1}{2}\sum_{i=1}^{n} \left|\vec{\omega} \times \vec{r}_i'\right|^2 \Delta m_i$$

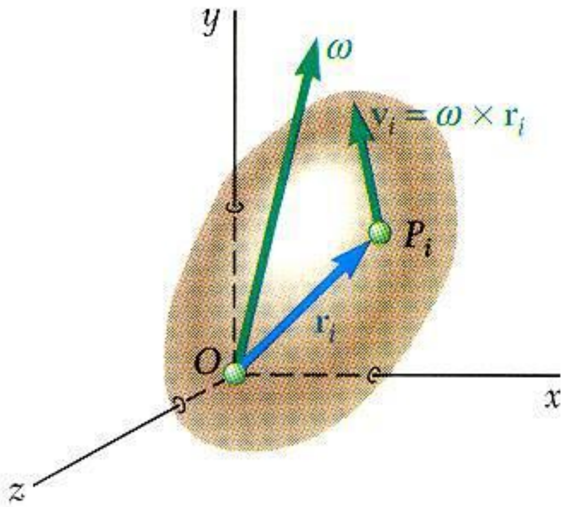
$$= \tfrac{1}{2} m\bar{v}^2 + \tfrac{1}{2}(\bar{I}_x\omega_x^2 + \bar{I}_y\omega_y^2 + \bar{I}_z\omega_z^2 - 2\bar{I}_{xy}\omega_x\omega_y$$
$$- 2\bar{I}_{yz}\omega_y\omega_z - 2\bar{I}_{zx}\omega_z\omega_x)$$

- If the axes correspond instantaneously with the principle axes,

$$T = \tfrac{1}{2} m\bar{v}^2 + \tfrac{1}{2}(\bar{I}_{x'}\omega_{x'}^2 + \bar{I}_{y'}\omega_{y'}^2 + \bar{I}_{z'}\omega_{z'}^2)$$

- With these results, the principles of work and energy and conservation of energy may be applied to the three-dimensional motion of a rigid body.

Kinetic Energy



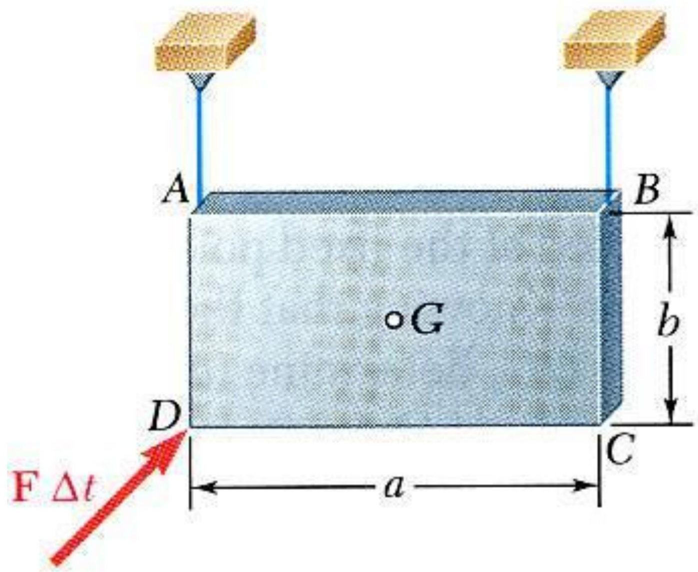
- Kinetic energy of a rigid body with a fixed point,

$$T = \tfrac{1}{2}\left(I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2 - 2I_{xy}\omega_x\omega_y - 2I_{yz}\omega_y\omega_z - 2I_{zx}\omega_z\omega_x\right)$$

- If the axes *Oxyz* correspond instantaneously with the principle axes *Ox'y'z'*,

$$T = \tfrac{1}{2}\left(I_{x'}\omega_{x'}^2 + I_{y'}\omega_{y'}^2 + I_{z'}\omega_{z'}^2\right)$$

Sample Problem 18.1



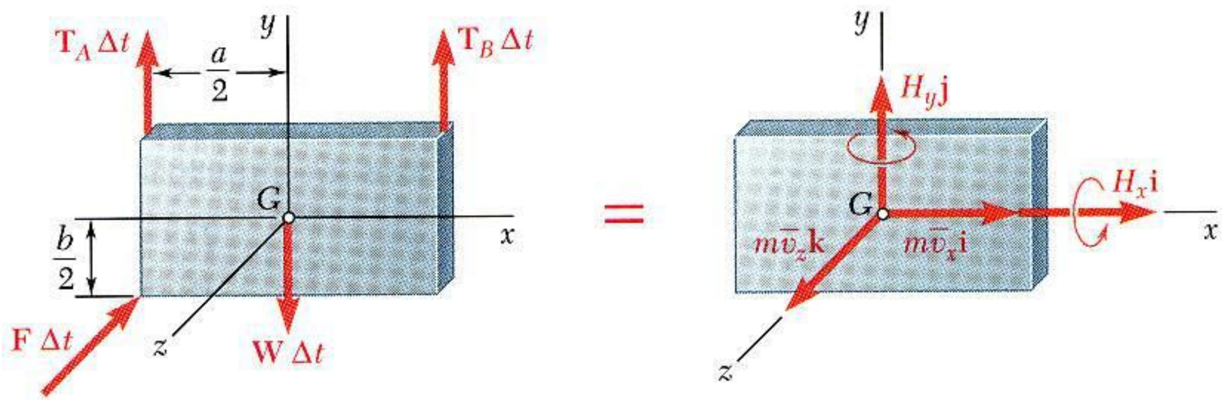
Rectangular plate of mass *m* that is suspended from two wires is hit at *D* in a direction perpendicular to the plate.

Immediately after the impact, determine *a*) the velocity of the mass center *G*, and *b*) the angular velocity of the plate.

<u>SOLUTION</u>:

- Apply the principle of impulse and momentum. Since the initial momenta is zero, the system of impulses must be equivalent to the final system of momenta.
- Assume that the supporting cables remain taut such that the vertical velocity and the rotation about an axis normal to the plate is zero.
- Principle of impulse and momentum yields to two equations for linear momentum and two equations for angular momentum.
- Solve for the two horizontal components of the linear and angular velocity vectors.

Sample Problem 18.1



SOLUTION:

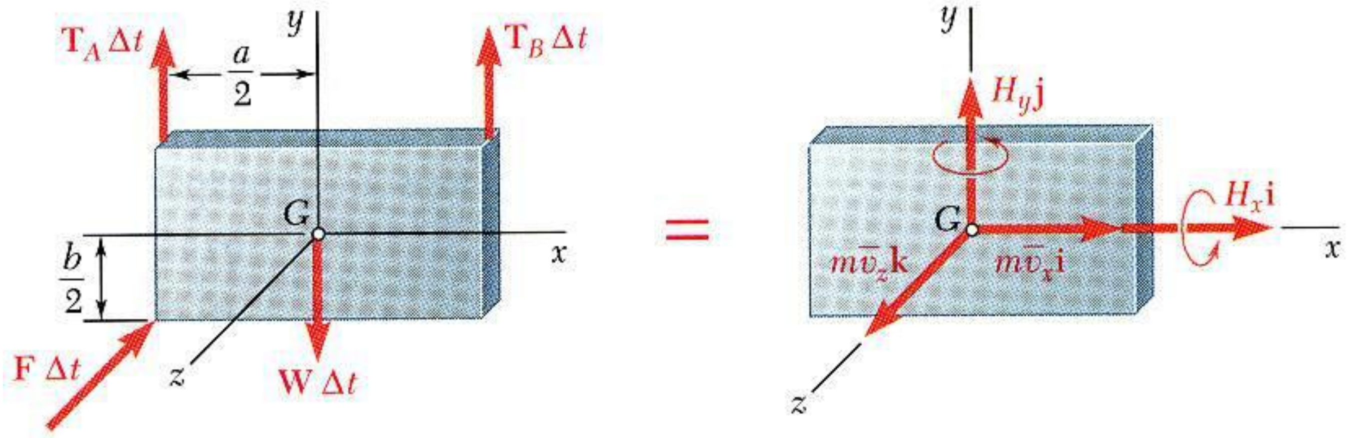
- Apply the principle of impulse and momentum. Since the initial momenta is zero, the system of impulses must be equivalent to the final system of momenta.
- Assume that the supporting cables remain taut such that the vertical velocity and the rotation about an axis normal to the plate is zero.

$$\vec{\bar{v}} = \bar{v}_x \vec{i} + v_z \vec{k} \qquad \vec{\omega} = \omega_x \vec{i} + \omega_y \vec{j}$$

Since the x, y, and z axes are principal axes of inertia,

$$\vec{H}_G = \bar{I}_x \omega_x \vec{i} + \bar{I}_y \omega_y \vec{j} = \tfrac{1}{12} m b^2 \omega_x \vec{i} + \tfrac{1}{12} m a^2 \omega_y \vec{j}$$

Sample Problem 18.1



- Principle of impulse and momentum yields two equations for linear momentum and two equations for angular momentum.

- Solve for the two horizontal components of the linear and angular velocity vectors.

$$0 = m v_x \qquad -F\Delta t = m\bar{v}_z$$

$$v_x = 0 \qquad \bar{v}_z = -F\Delta t/m$$

$$\boxed{\vec{\bar{v}} = -(F\Delta t/m)\vec{k}}$$

$$\tfrac{1}{2} bF\Delta t = H_x = \tfrac{1}{12} mb^2 \omega_x$$

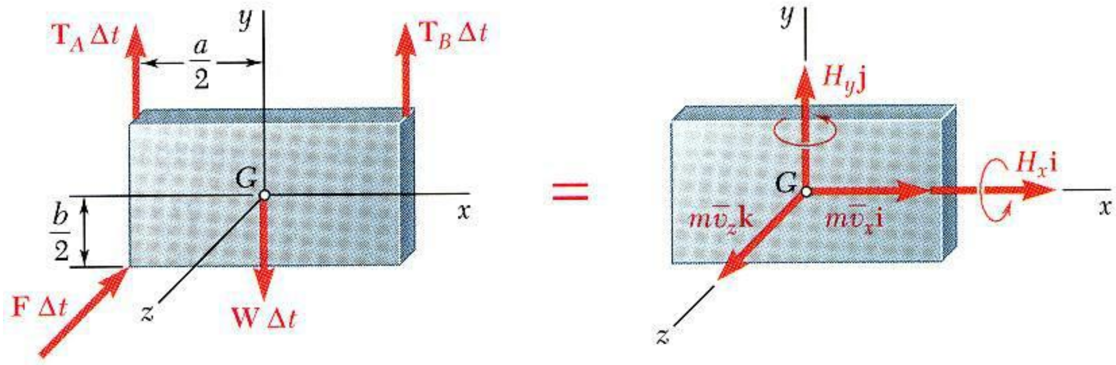
$$\omega_x = 6F\Delta t/mb$$

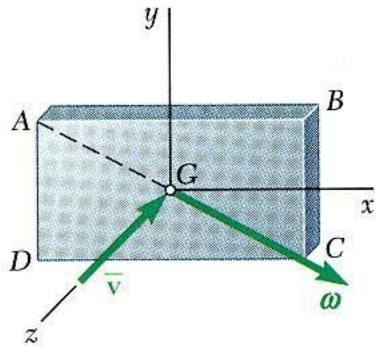
$$-\tfrac{1}{2} aF\Delta t = H_y = \tfrac{1}{12} ma^2 \omega_y$$

$$\omega_y = -(6F\Delta t/ma)$$

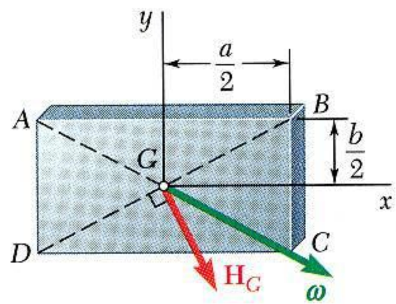
$$\boxed{\vec{\omega} = \frac{6F\Delta t}{mab}\left(a\vec{i} + b\vec{j}\right)}$$

Sample Problem 18.1





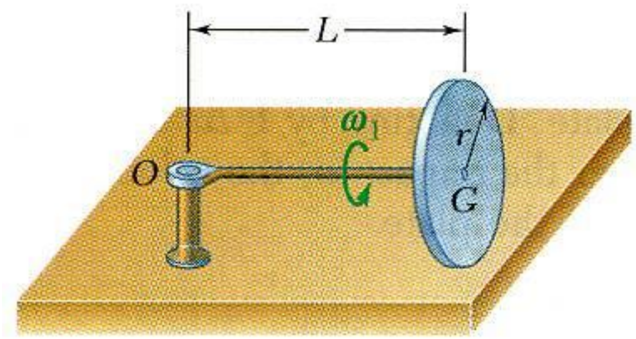
$$\vec{v} - (F\Delta t/m)\vec{k}$$



$$\vec{\omega} = \frac{6F\Delta t}{mab}\left(a\vec{i} + b\vec{j}\right)$$

$$\vec{H}_G = \tfrac{1}{12}mb^2\omega_x\vec{i} + \tfrac{1}{12}ma^2\omega_y\vec{j}$$

Sample Problem 18.2



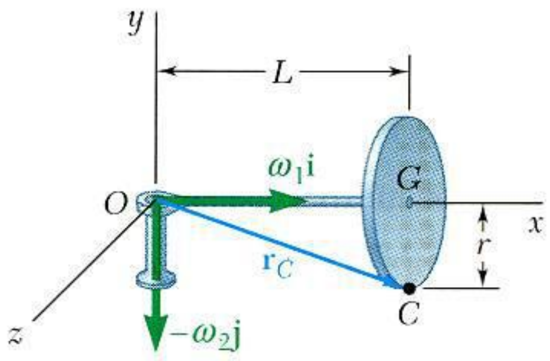
A homogeneous disk of mass *m* is mounted on an axle *OG* of negligible mass. The disk rotates counter-clockwise at the rate ω_1 about *OG*.

Determine: *a*) the angular velocity of the disk, *b*) its angular momentum about *O*, *c*) its kinetic energy, and d) the vector and couple at *G* equivalent to the momenta of the particles of the disk.

SOLUTION:

- The disk rotates about the vertical axis through *O* as well as about *OG*. Combine the rotation components for the angular velocity of the disk.
- Compute the angular momentum of the disk using principle axes of inertia and noting that *O* is a fixed point.
- The kinetic energy is computed from the angular velocity and moments of inertia.
- The vector and couple at *G* are also computed from the angular velocity and moments of inertia.

Sample Problem 18.2



SOLUTION:

- The disk rotates about the vertical axis through *O* as well as about *OG*. Combine the rotation components for the angular velocity of the disk.

$$\vec{\omega} = \omega_1 \vec{i} + \omega_2 \vec{j}$$

Noting that the velocity at *C* is zero,

$$\vec{v}_C = \vec{\omega} \times \vec{r}_C = 0$$

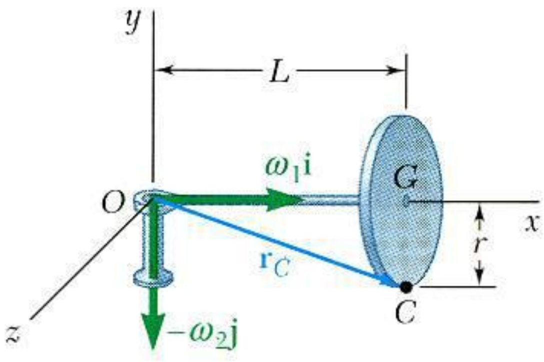
$$0 = \left(\omega_1 \vec{i} + \omega_2 \vec{j}\right) \times \left(L\vec{i} - r\vec{j}\right)$$

$$= \left(L\omega_2 - r\omega_1\right)\vec{k}$$

$$\omega_2 = r\omega_1 / L$$

$$\boxed{\vec{\omega} = \omega_1 \vec{i} - \left(r\omega_1 / L\right)\vec{j}}$$

Sample Problem 18.2



$$\vec{\omega} = \omega_1 \vec{i} - (r\omega_1/L)\vec{j}$$

- Compute the angular momentum of the disk using principle axes of inertia and noting that O is a fixed point.

$$\vec{H}_O = I_x \omega_x \vec{i} + I_y \omega_y \vec{j} + I_z \omega_z \vec{k}$$

$$H_x = I_x \omega_x = \left(\tfrac{1}{2} m r^2\right) \omega_1$$

$$H_y = I_y \omega_y = \left(mL^2 + \tfrac{1}{4} m r^2\right)\left(-r\omega_1/L\right)$$

$$H_z = I_z \omega_z = \left(mL^2 + \tfrac{1}{4} m r^2\right) 0 = 0$$

$$\vec{H}_O = \tfrac{1}{2} m r^2 \omega_1 \vec{i} - m\left(L^2 + \tfrac{1}{4} r^2\right)\left(r\omega_1/L\right)\vec{j}$$

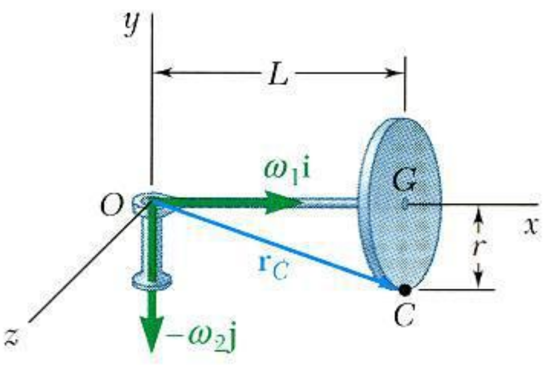
- The kinetic energy is computed from the angular velocity and moments of inertia.

$$T = \tfrac{1}{2}\left(I_x \omega_x^2 + I_y \omega_y^2 + I_z \omega_z^2\right)$$

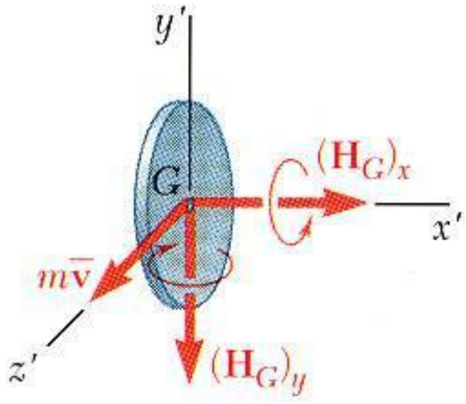
$$= \tfrac{1}{2}\left[m r^2 \omega_1^2 + m\left(L^2 + \tfrac{1}{4} r^2\right)\left(-r\omega_1/L\right)^2\right]$$

$$T = \tfrac{1}{8} m r^2 \left(6 + \frac{r^2}{L^2}\right)\omega_1^2$$

Sample Problem 18.2



$$\vec{\omega} = \omega_1 \vec{i} - (r\omega_1/L)\vec{j}$$



- The vector and couple at *G* are also computed from the angular velocity and moments of inertia.

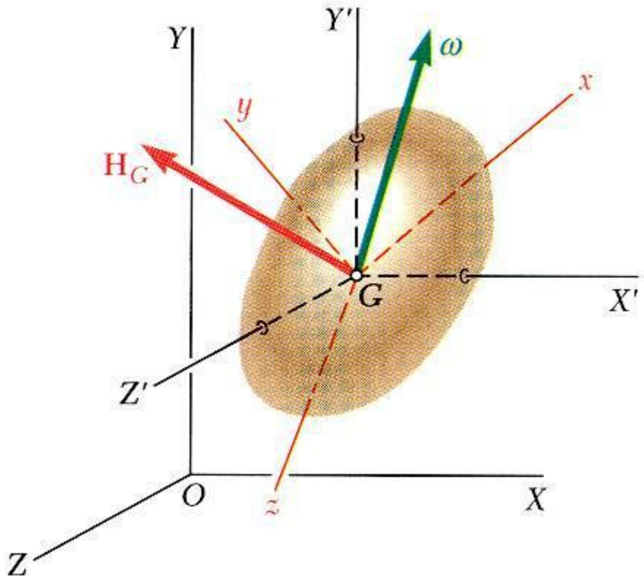
$$m\vec{\bar{v}} = mr\omega_1\vec{k}$$

$$\vec{H}_G = \bar{I}_{x'}\omega_x\vec{i} + \bar{I}_{y'}\omega_y\vec{j} + \bar{I}_{z'}\omega_z\vec{k}$$

$$= \tfrac{1}{2}mr^2\omega_1\vec{i} + \tfrac{1}{4}mr^2(-r\omega/L)\vec{j}$$

$$\vec{H}_G = \tfrac{1}{2}mr^2\omega_1\left(\vec{i} - \frac{r}{2L}\vec{j}\right)$$

Motion of a Rigid Body in Three Dimensions



$$\sum \vec{F} = m\vec{a}$$

$$\sum \vec{M} = \dot{\vec{H}}_G$$

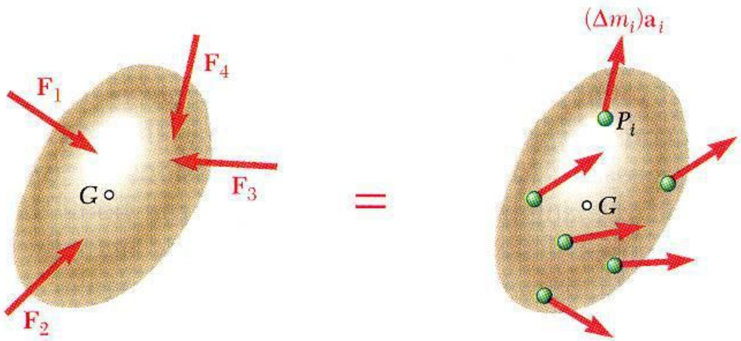
- Angular momentum and its rate of change are taken with respect to centroidal axes *GX'Y'Z'* of fixed orientation.
- Transformation of $\vec{\omega}$ into $\vec{H}_G$ is independent of the system of coordinate axes.
- Convenient to use body fixed axes *Gxyz* where moments and products of inertia are not time dependent.
- Define rate of change of change of $\vec{H}_G$ with respect to the rotating frame,

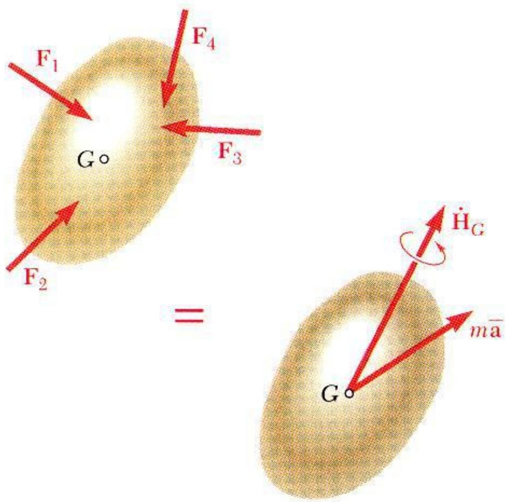
$$\left(\dot{\vec{H}}_G\right)_{Gxyz} = \dot{H}_x\vec{i} + \dot{H}_y\vec{j} + \dot{H}_z\vec{k}$$

Then,

$$\dot{\vec{H}}_G = \left(\dot{\vec{H}}_G\right)_{Gxyz} + \vec{\Omega}\times\vec{H}_G \qquad \vec{\Omega} = \vec{\omega}$$

Euler's Eqs of Motion & D'Alembert's Principle





- With $\vec{\Omega} = \vec{\omega}$ and $Gxyz$ chosen to correspond to the principal axes of inertia,

$$\sum \vec{M}_G = \left(\dot{\vec{H}}_G\right)_{Gxyz} + \vec{\Omega} \times \vec{H}_G$$

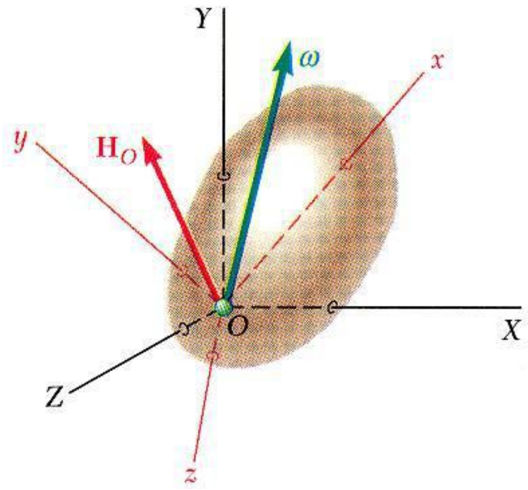
Euler's Equations:

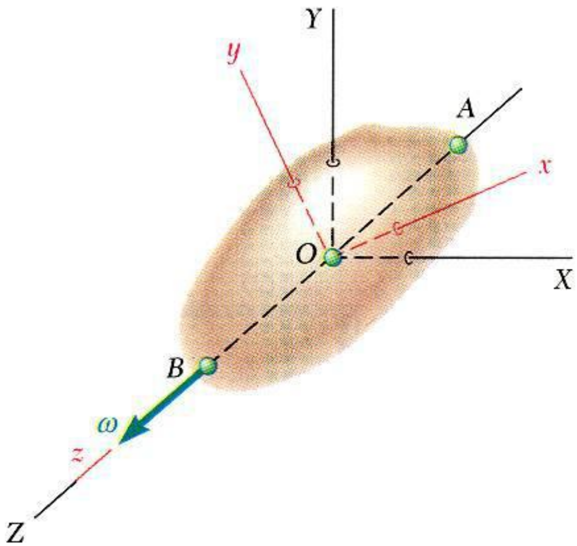
$$\sum M_x = \bar{I}_x \dot{\omega}_x - \left(\bar{I}_y - \bar{I}_z\right)\omega_y \omega_z$$
$$\sum M_y = \bar{I}_y \dot{\omega}_y - \left(\bar{I}_z - \bar{I}_x\right)\omega_z \omega_x$$
$$\sum M_z = \bar{I}_z \dot{\omega}_z - \left(\bar{I}_x - \bar{I}_y\right)\omega_x \omega_y$$

- System of external forces and effective forces are equivalent for general three dimensional motion.

- System of external forces are equivalent to the vector and couple, $m\vec{\bar{a}}$ and $\dot{\vec{H}}_G$.

Motion About a Fixed Point or a Fixed Axis





- For a rigid body rotation around a fixed point,

$$\sum \vec{M}_O = \dot{\vec{H}}_O = \left(\dot{\vec{H}}_O\right)_{Oxyz} + \vec{\Omega} \times \vec{H}_O$$

- For a rigid body rotation around a fixed axis,

$$H_x = -I_{xz}\omega \qquad H_y = -I_{yz}\omega \qquad H_z = -I_z\omega$$

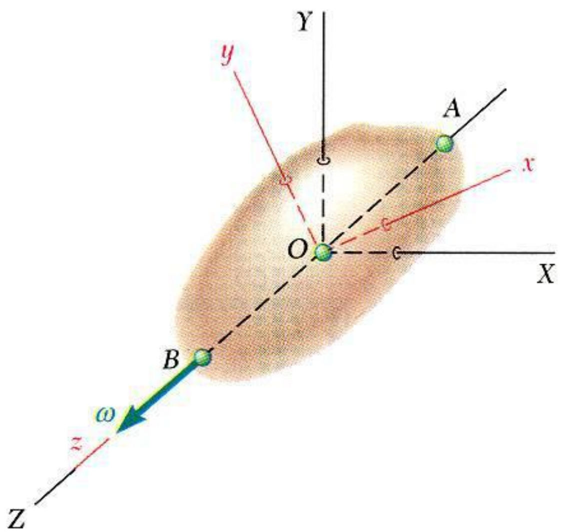
$$\begin{aligned}\sum \vec{M}_O &= \left(\dot{\vec{H}}_O\right)_{Oxyz} + \vec{\omega} \times \vec{H}_O \\ &= \left(-I_{xz}\vec{i} - I_{yz}\vec{j} + I_z\vec{k}\right)\dot{\omega} \\ &\quad + \omega\vec{k} \times \left(-I_{xz}\vec{i} - I_{yz}\vec{j} + I_z\vec{k}\right)\omega \\ &= \left(-I_{xz}\vec{i} - I_{yz}\vec{j} + I_z\vec{k}\right)\alpha + \left(-I_{xz}\vec{j} + I_{yz}\vec{i}\right)\omega^2\end{aligned}$$

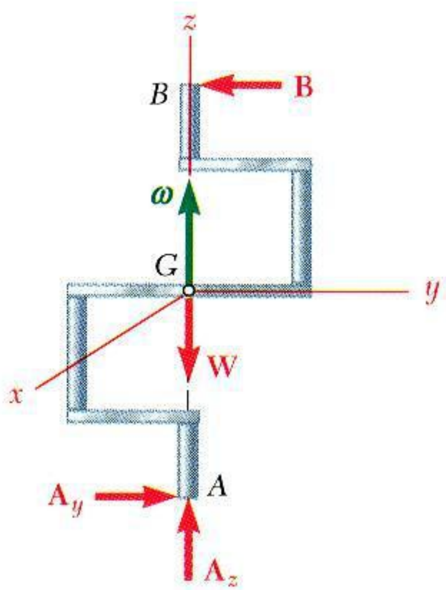
$$\sum M_x = -I_{xz}\alpha + I_{yz}\omega^2$$

$$\sum M_y = -I_{yz}\alpha + I_{xz}\omega^2$$

$$\sum M_z = I_z\alpha$$

Rotation About a Fixed Axis





- For a rigid body rotation around a fixed axis,

$$\sum M_x = -I_{xz}\alpha + I_{yz}\omega^2$$

$$\sum M_y = -I_{yz}\alpha + I_{xz}\omega^2$$

$$\sum M_z = I_z\alpha$$

- If symmetrical with respect to the xy plane,

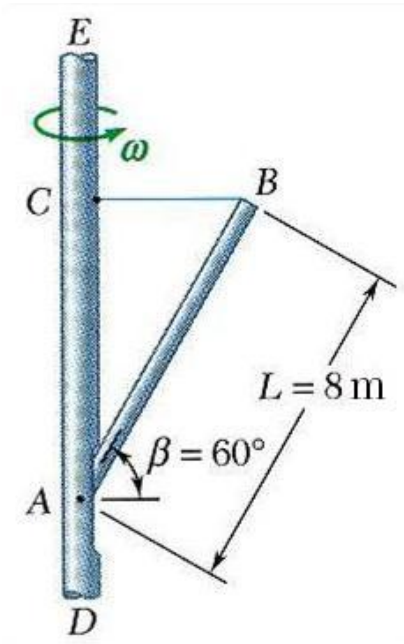
$$\sum M_x = 0 \quad \sum M_y = 0 \quad \sum M_z = I_z\alpha$$

- If not symmetrical, the sum of external moments will not be zero, even if $\alpha = 0$,

$$\sum M_x = I_{yz}\omega^2 \quad \sum M_y = I_{xz}\omega^2 \quad \sum M_z = 0$$

- A rotating shaft requires both static $(\omega = 0)$ and dynamic $(\omega \neq 0)$ balancing to avoid excessive vibration and bearing reactions.

Sample Problem 18.3



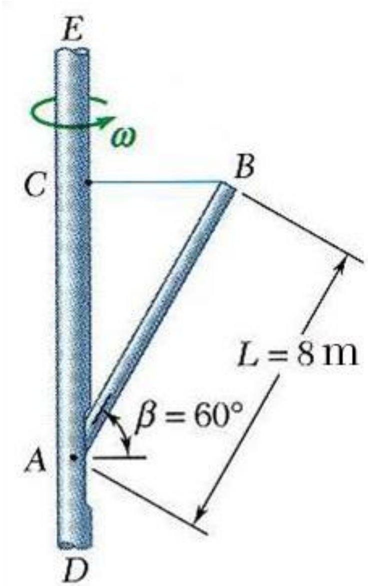
Rod *AB* with weight *W* = 40 N is pinned at *A* to a vertical axle which rotates with constant angular velocity ω = 15 rad/s. The rod position is maintained by a horizontal wire *BC*.

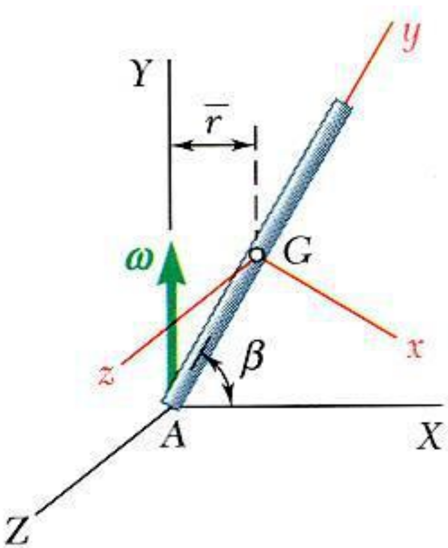
Determine the tension in the wire and the reaction at *A*.

SOLUTION:

- Evaluate the system of effective forces by reducing them to a vector $m\vec{\bar{a}}$ attached at *G* and couple $\dot{\vec{H}}_G$.
- Expressing that the system of external forces is equivalent to the system of effective forces, write vector expressions for the sum of moments about *A* and the summation of forces.
- Solve for the wire tension and the reactions at *A*.

Sample Problem 18.3





SOLUTION:

- Evaluate the system of effective forces by reducing them to a vector $m\vec{\bar{a}}$ attached at G and couple $\vec{H}_G$.

$$\vec{\bar{a}} = \vec{a}_n = -\bar{r}\omega^2\vec{I} = -\left(\tfrac{1}{2}L\cos\beta\right)\omega^2\vec{I}$$

$$= -\left(450\ \text{m/s}^2\right)\vec{I}$$

$$m\vec{\bar{a}} = \frac{40}{g}(-450) = -(1800\ \text{N})\vec{I}$$

$$\vec{H}_G = \bar{I}_x\omega_x\vec{i} + \bar{I}_y\omega_y\vec{j} + \bar{I}_z\omega_z\vec{k}$$

$$\bar{I}_x = \tfrac{1}{2}mL^2 \qquad \bar{I}_y = 0 \qquad \bar{I}_z = \tfrac{1}{2}mL^2$$

$$\omega_x = -\omega\cos\beta \qquad \omega_y = \omega\sin\beta \qquad \omega_z = 0$$

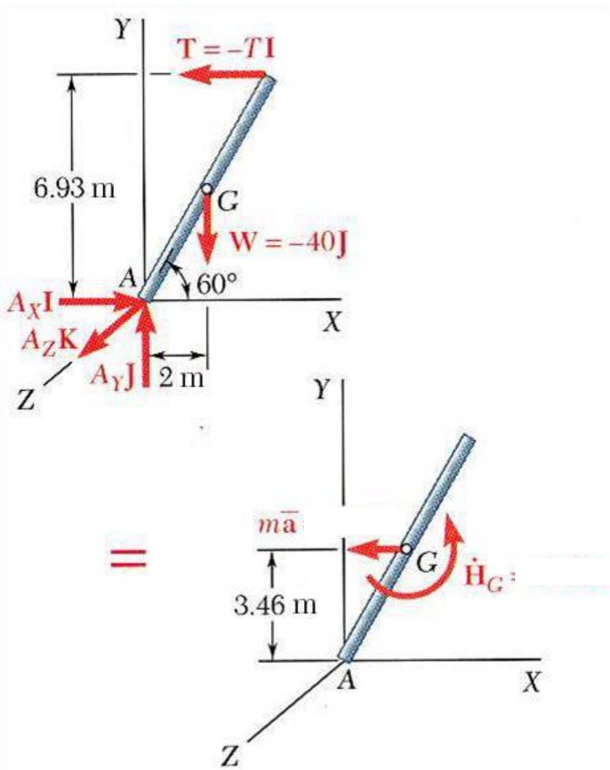
$$\vec{H}_G = -\tfrac{1}{12}mL^2\omega\cos\beta\,\vec{i}$$

$$\dot{\vec{H}}_G = \left(\dot{\vec{H}}_G\right)_{Gxyz} + \vec{\omega}\times\vec{H}_G$$

$$= 0 + \left(-\omega\cos\beta\,\vec{i} + \omega\sin\beta\,\vec{j}\right)\times\left(\tfrac{1}{12}mL^2\omega\cos\beta\,\vec{i}\right)$$

$$= \tfrac{1}{12}mL^2\omega^2\sin\beta\cos\beta\,\vec{k} = (2078.4\ \text{N}\cdot\text{m})\vec{k}$$

Sample Problem 18.3



- Expressing that the system of external forces is equivalent to the system of effective forces, write vector expressions for the sum of moments about *A* and the summation of forces.

$$\sum \vec{M}_A = \sum \left(\vec{M}_A\right)_{eff}$$

$$6.93\vec{J} \times \left(-T\vec{I}\right) + 2\vec{I} \times \left(-40\vec{J}\right) = 3.46\vec{J} \times \left(-1800\vec{I}\right) + 2078.4\vec{K}$$

$$(6.93T - 80)\vec{K} = (6228 + 2078.4)\vec{K}$$

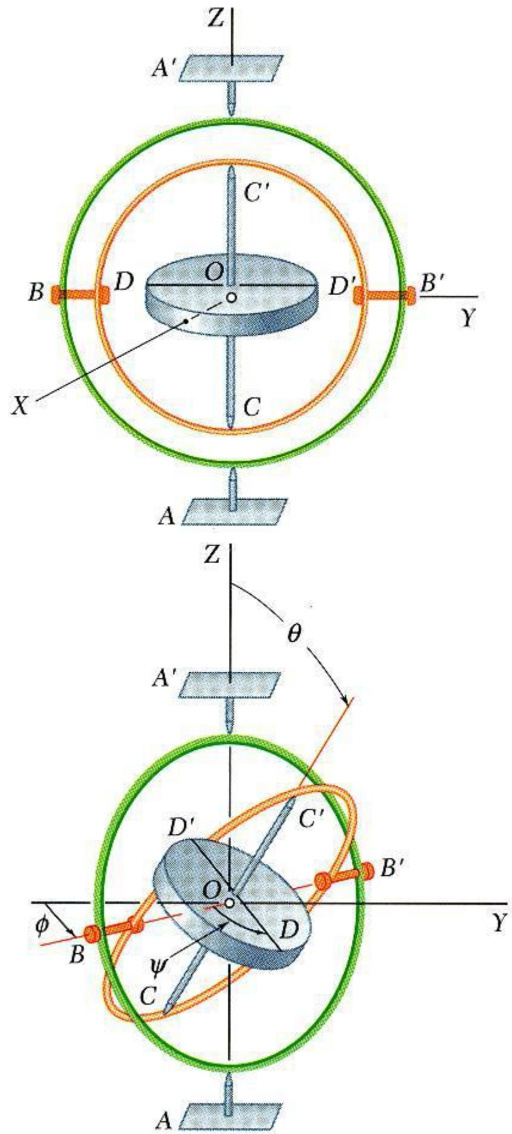
$$\boxed{T = 1210 \text{ N}}$$

$$\sum \vec{F} = \sum \left(\vec{F}\right)_{eff}$$

$$A_X\vec{I} + A_Y\vec{J} + A_Z\vec{K} - 1210\vec{I} - 40\vec{J} = -1800\vec{I}$$

$$\boxed{\vec{A} = -(590 \text{ N})\vec{I} + (40 \text{ N})\vec{J}}$$

Motion of a Gyroscope. Eulerian Angles



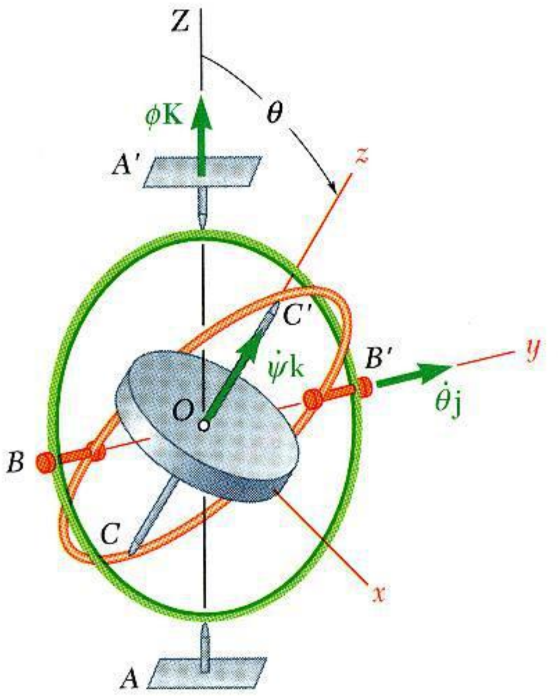
- A gyroscope consists of a rotor with its mass center fixed in space but which can spin freely about its geometric axis and assume any orientation.
- From a reference position with gimbals and a reference diameter of the rotor aligned, the gyroscope may be brought to any orientation through a succession of three steps:
  - a) rotation of outer gimbal through *j* about *AA'*,
  - b) rotation of inner gimbal through *q* about
  - c) rotation of the rotor through *y* about *CC'*.
- φ, θ, and ψ are called the *Eulerian Angles* and

$$\dot{\phi} = \text{rate of precession}$$

$$\dot{\theta} = \text{rate of nutation}$$

$$\dot{\Psi} = \text{rate of spin}$$

Motion of a Gyroscope. Eulerian Angles



- The angular velocity of the gyroscope,

$$\vec{\omega} = \dot{\phi}\vec{K} + \dot{\theta}\vec{j} + \dot{\Psi}\vec{k}$$

with $\vec{K} = -\sin\theta\,\vec{i} + \cos\theta\,\vec{j}$

$$\vec{\omega} = -\dot{\phi}\sin\theta\,\vec{i} + \dot{\theta}\vec{j} + \left(\dot{\Psi} + \dot{\phi}\cos\theta\right)\vec{k}$$

- Equation of motion,

$$\sum \vec{M}_O = \left(\dot{\vec{H}}_O\right)_{Oxyz} + \vec{\Omega}\times\vec{H}_O$$

$$\vec{H}_O = -I'\dot{\phi}\sin\theta\,\vec{i} + I'\dot{\theta}\vec{j} + I\left(\dot{\Psi} + \dot{\phi}\cos\theta\right)\vec{k}$$

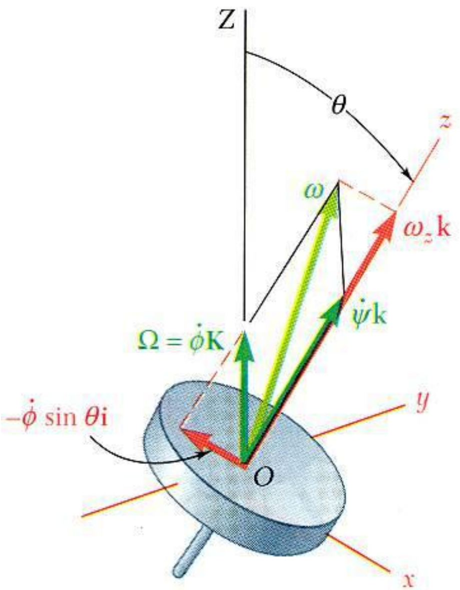
$$\vec{\Omega} = \dot{\phi}\vec{K} + \dot{\theta}\vec{j}$$

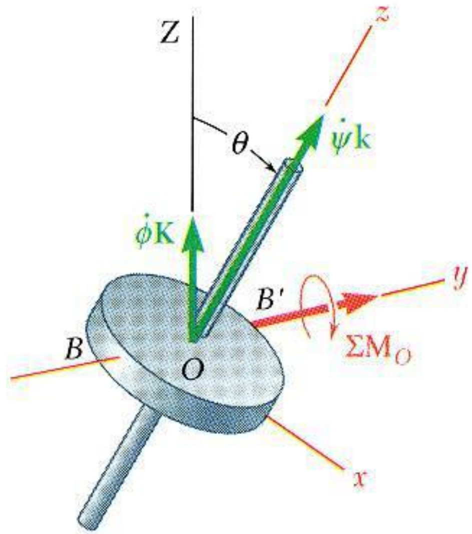
$$\sum M_x = -I'\left(\ddot{\phi}\sin\theta + 2\dot{\theta}\dot{\phi}\cos\theta\right) + I\dot{\theta}\left(\dot{\Psi} + \dot{\phi}\cos\theta\right)$$

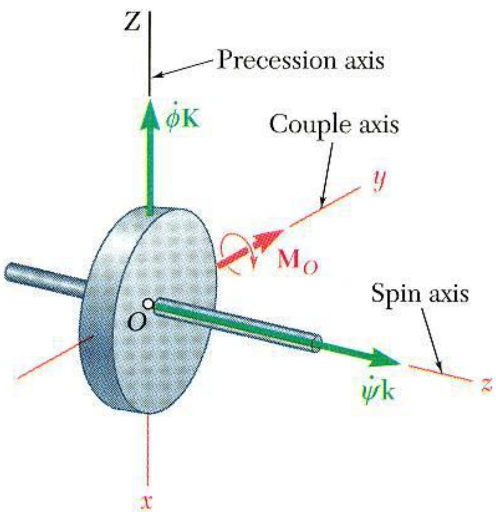
$$\sum M_y = I'\left(\ddot{\theta} - \dot{\phi}^2\sin\theta\cos\theta\right) + I\dot{\phi}\sin\theta\left(\dot{\Psi} + \dot{\phi}\cos\theta\right)$$

$$\sum M_z = I\frac{d}{dt}\left(\dot{\Psi} + \dot{\phi}\cos\theta\right)$$

Steady Precession of a Gyroscope







Steady precession,

$\theta, \dot{\phi}, \dot{\psi}$ are constant

$$\vec{\omega} = -\dot{\phi}\sin\theta\,\vec{i} + \omega_z\vec{k}$$

$$\vec{H}_O = -I'\dot{\phi}\sin\theta\,\vec{i} + I\omega_z\vec{k}$$

$$\vec{\Omega} = -\dot{\phi}\sin\theta\,\vec{i} + \dot{\phi}\cos\theta\,\vec{k}$$

$$\sum\vec{M}_O = \vec{\Omega}\times\vec{H}_O$$

$$= \left(I\omega_z - I'\dot{\phi}\cos\theta\right)\dot{\phi}\sin\theta\,\vec{j}$$

Couple is applied about an axis perpendicular to the precession and spin axes

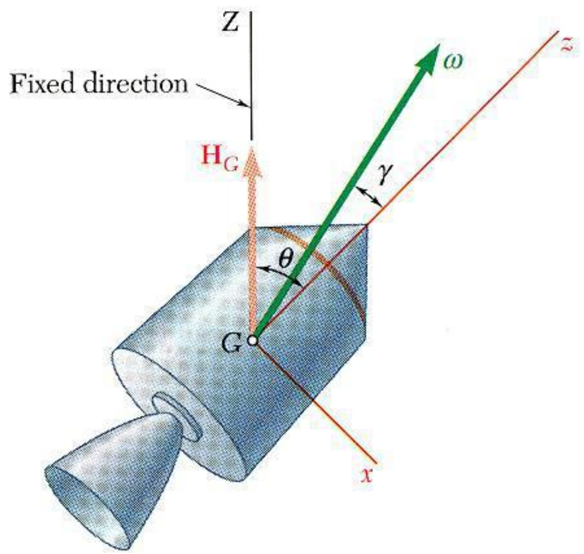
When the precession and spin axis are at a right angle,

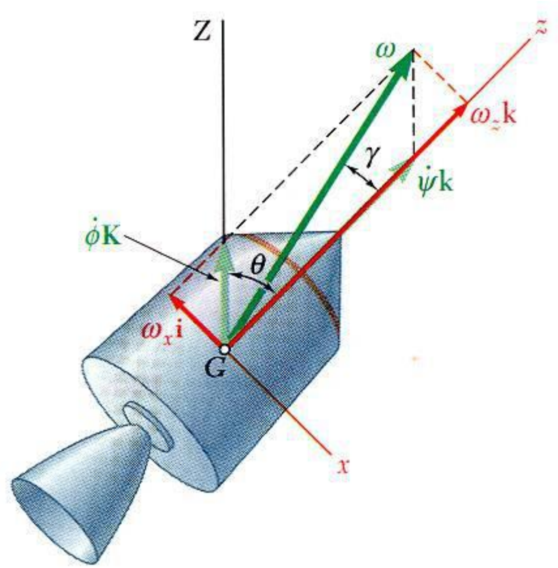
$$\theta = 90°$$

$$\sum\vec{M}_O = I\dot{\Psi}\dot{\phi}\,\vec{j}$$

Gyroscope will precess about an axis perpendicular to both the spin axis and couple axis.

Motion of an Axisymmetrical Body Under No Force





- Consider motion about its mass center of an axisymmetrical body under no force but its own weight, e.g., projectiles, satellites, and space craft.

$$\dot{\vec{H}}_G = 0 \qquad \vec{H}_G = \text{constant}$$

- Define the Z axis to be aligned with $\vec{H}_G$ and z in a rotating axes system along the axis of symmetry. The x axis is chosen to lie in the Zz plane.

$$H_x = -H_G \sin\theta = I'\omega_x \qquad \omega_x = -\frac{H_G \sin\theta}{I'}$$

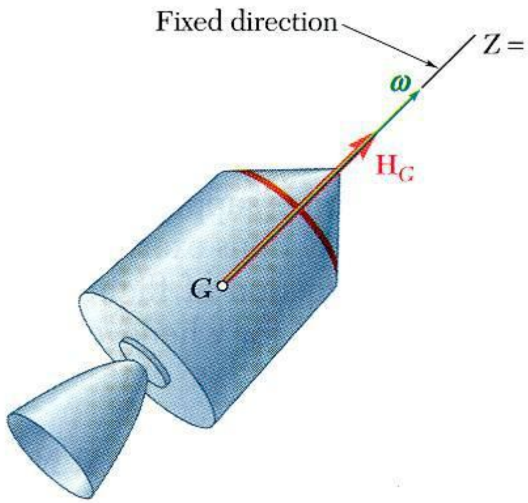
$$H_y = 0 = I'\omega_y \qquad \omega_y = 0$$

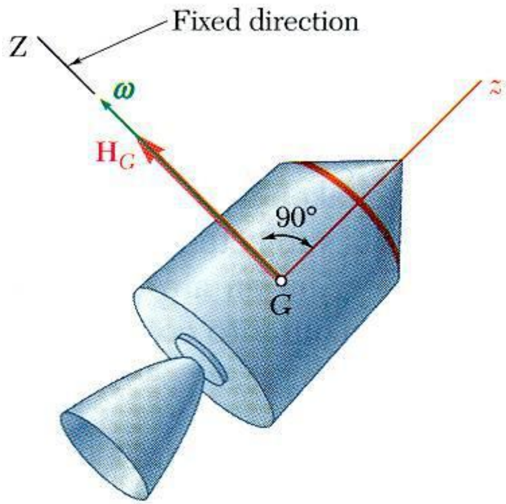
$$H_z = H_G \cos\theta = I\omega_z \qquad \omega_z = \frac{H_G \cos\theta}{I}$$

- θ = constant and body is in steady precession.

- Note: $-\dfrac{\omega_x}{\omega_z} = \tan\gamma = \dfrac{I}{I'}\tan\theta$

Motion of an Axisymmetrical Body Under No Force





Two cases of motion of an axisymmetrical body which under no force which involve no precession:

- Body set to spin about its axis of symmetry,

  $$\omega_x = H_x = 0$$

  $$\vec{\omega} \text{ and } \vec{H}_G \text{ are aligned}$$

  and body keeps spinning about its axis of symmetry.

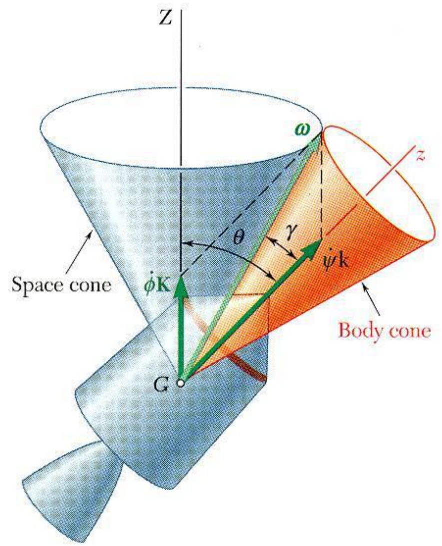
- Body is set to spin about its transverse axis,

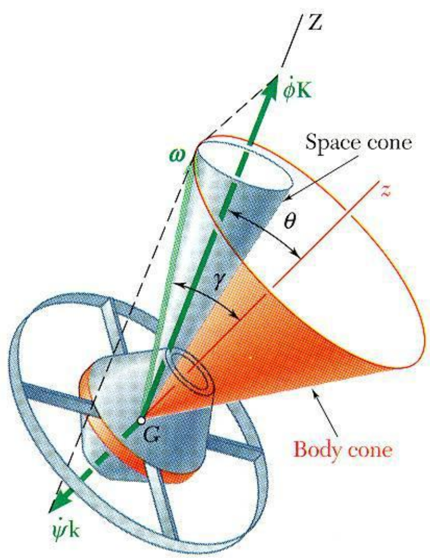
  $$\omega_z = H_z = 0$$

  $$\vec{\omega} \text{ and } \vec{H}_G \text{ are aligned}$$

  and body keeps spinning about the given transverse axis.

Motion of an Axisymmetrical Body Under No Force





The motion of a body about a fixed point (or its mass center) can be represented by the motion of a body cone rolling on a space cone. In the case of steady precession the two cones are circular.

- *I* < *I'*. Case of an elongated body. $\gamma < \theta$ and the vector ω lies inside the angle *ZGz*. The space cone and body cone are tangent externally; the spin and precession are both counterclockwise from the positive *z* axis. The precession is said to be *direct*.

- *I* > *I'*. Case of a flattened body. $\gamma > \theta$ and the vector ω lies outside the angle *ZGz*. The space cone is inside the body cone; the spin and precession have opposite senses. The precession is said to be *retrograde*.