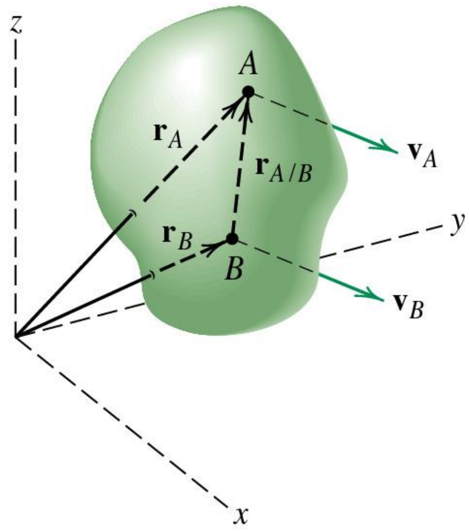
THREE-DIMENSIONAL KINEMATICS OF RIGID BODIES

1. TRANSLATION



$$\vec{r}_A = \vec{r}_B + \vec{r}_{A/B}$$

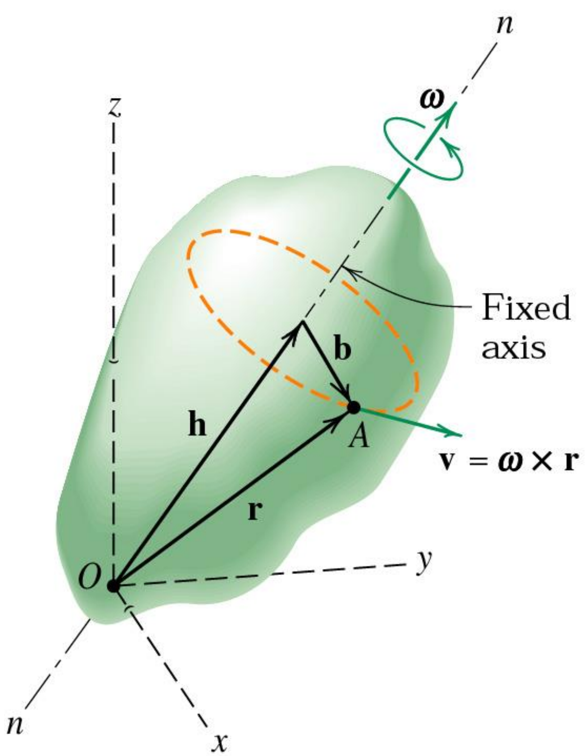
$$\vec{v}_A = \vec{v}_B$$

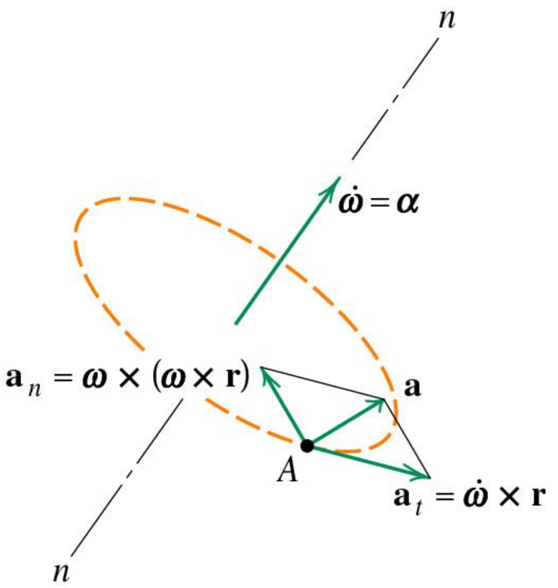
$$\vec{a}_A = \vec{a}_B$$

$\vec{r}_{A/B}$ remains constant and therefore its time derivative is zero.

Thus, all points in the body have the same velocity and the same acceleration.

2. FIXED-AXIS ROTATION





Any point such as A which is not on the axis moves in a circular arc in a plane normal to the axis and has a velocity and an acceleration

$$\vec{v} = \vec{\omega} \times \vec{r}$$

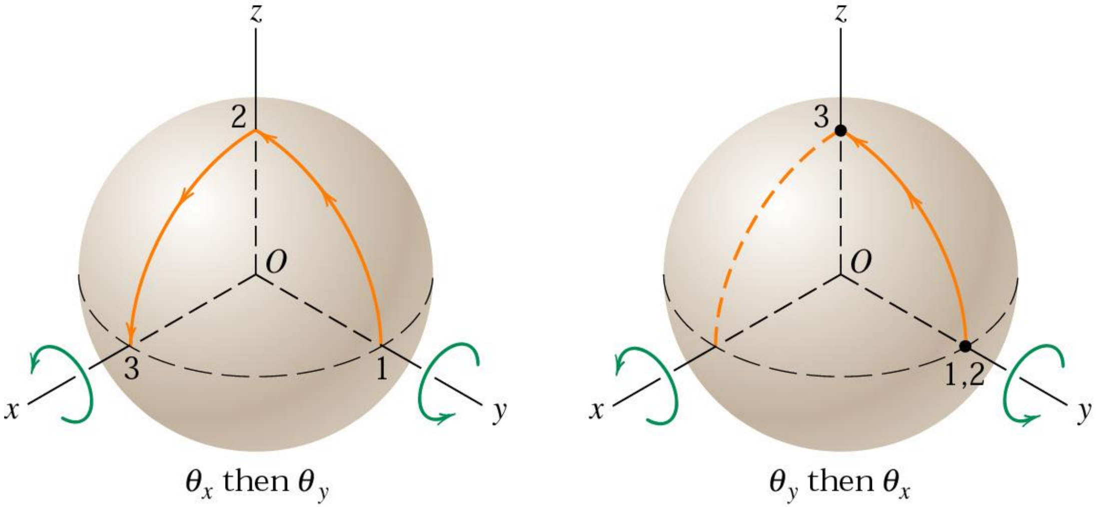
$$\vec{a} = \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times \left(\vec{\omega} \times \vec{r} \right)$$

3. ROTATION ABOUT A FIXED POINT

When a body rotates about a fixed point, the angular velocity vector no longer remains fixed in direction, and this change calls for a more general concept of rotation.

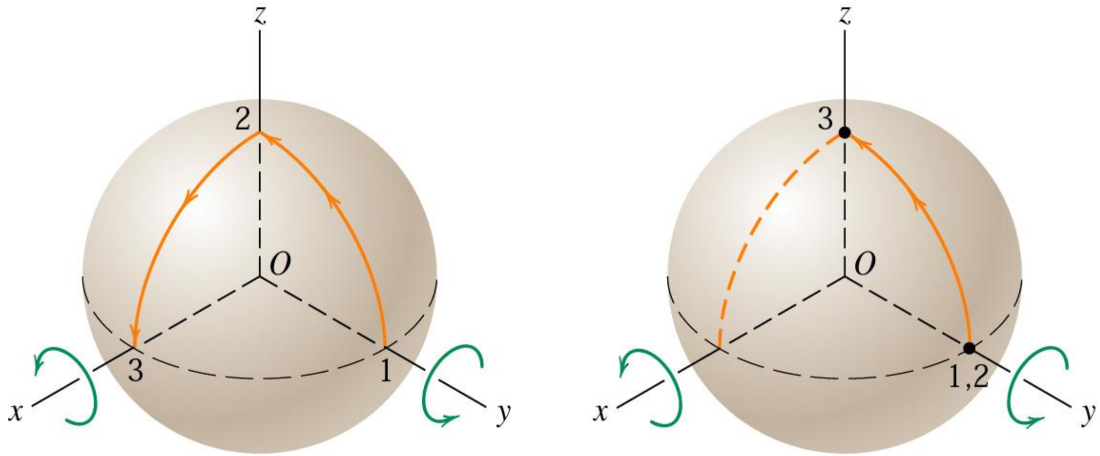
Rotation and Proper Vectors

Consider a solid sphere which is cut from a rigid body confined to rotate about the fixed point P. The *x-y-z* axis here are taken as fixed in space and do not rotate with the body.



(a) (b)

In part (a) of the figure, two successive 90º rotations of the sphere about, first, the x-axis and second, the y axis result in the motion of a point which is initially on the y-axis in position 1, to positions 2 and 3, successively.



θ_x then θ_y

(*a*)

θ_y then θ_x

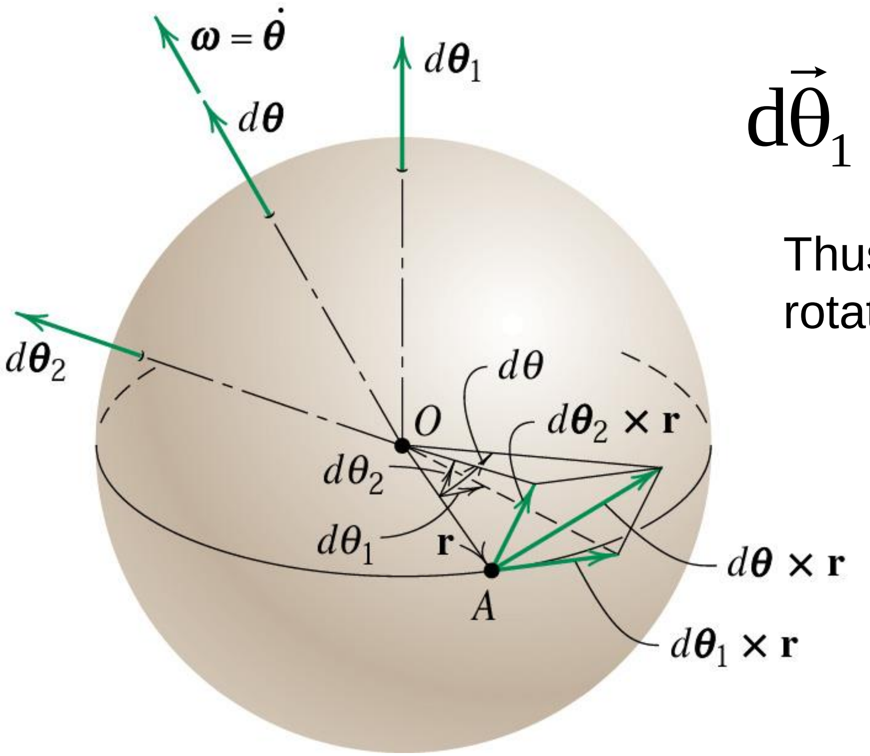
(*b*)

On the other hand, if the order of the rotations is reversed, the point undergoes no motion during the *y*-rotation but moves to point 3 during the 90º rotation about *x*-axis. Thus, the two cases do not produce the same final position, and it is evident from this one special example that finite rotations do not generally obey the parallelogram law of vector addition and are not commutative. (a+b≠b+a)

Thus, finite rotations may not be treated as proper vectors.

Infinitesimal rotations do obey the parallelogram law of vector addition.

Infinitesimal rotations are $d\theta_1$ and $d\theta_1$



As a result of $d\theta_1$ and $d\theta_2$, point A has a displacement

$$d\vec{\theta}_1 \times \vec{r} + d\vec{\theta}_2 \times \vec{r} = \left(d\vec{\theta}_1 + d\vec{\theta}_2\right) \times \vec{r}$$

Thus, two rotations are equivalent to the single rotation

$$d\vec{\theta} = d\vec{\theta}_1 + d\vec{\theta}_2$$

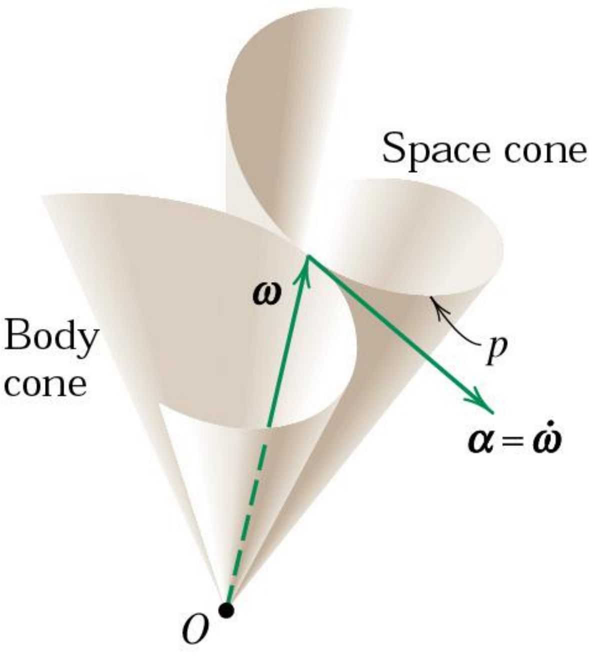
$$\dot{\vec{\theta}}_1 = \vec{\omega}_1 \qquad \dot{\vec{\theta}}_2 = \vec{\omega}_2$$

$$\vec{\omega} = \dot{\vec{\theta}} = \vec{\omega}_1 + \vec{\omega}_2$$

Angular Acceleration

The angular acceleration α of rigid body in three-dimensional motion is the time derivative of its angular velocity

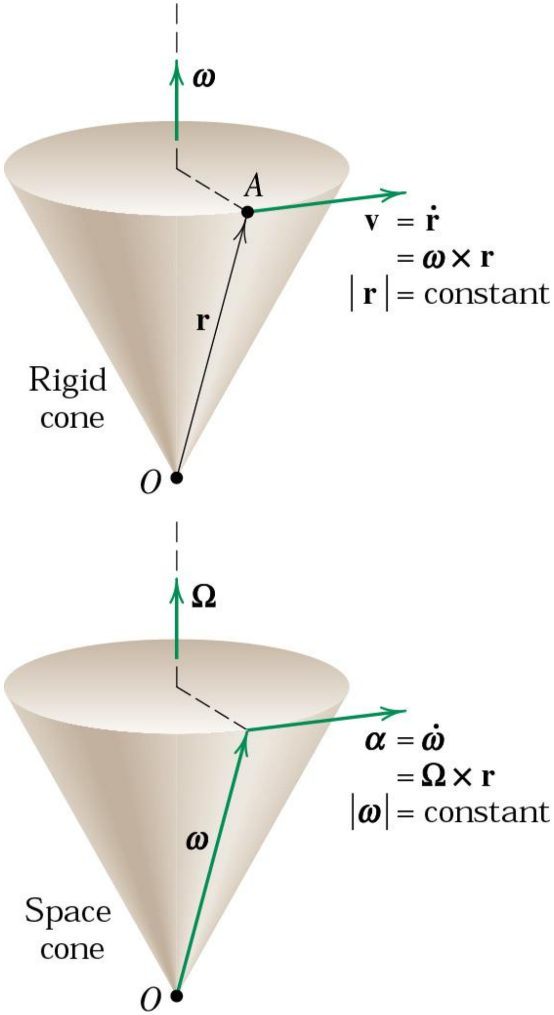
$$\vec{\alpha} = \dot{\vec{\omega}}$$



In contrast to the case of rotation in a single plane where the scalar a measures only the change in magnitude of the angular velocity, in three-dimensional motion the vector α reflects the change in direction as well as its change in magnitude.

Thus in the figure where the tip of the angular velocity vector ω follows the space curve p and changes in both magnitude and direction, the angular acceleration a becomes α vector tangent to this curve in the direction of the change in ω.

When the magnitude of ω remains constant, the angular acceleration α is normal to ω. For this case, if we let Ω stand for the angular velocity with the vector ω itself rotates (precesses) as it forms the space cone, the angular acceleration may be written



$$\vec{\alpha} = \vec{\Omega} \times \vec{\omega}$$

The upper part of the figure relates the velocity of point A on a rigid body to its position vector from O and the angular velocity of the body. The vectors α, ω, and Ω in the lower figure bear exactly the same relationship to each other as do vectors **v**, **r** and **ω** in the upper figure.

The one difference between the case of rotation about a fixed axis and rotation about a fixed point lies in the fact that for rotation about a fixed point, the angular acceleration

$$\vec{\alpha} = \dot{\vec{\omega}}$$

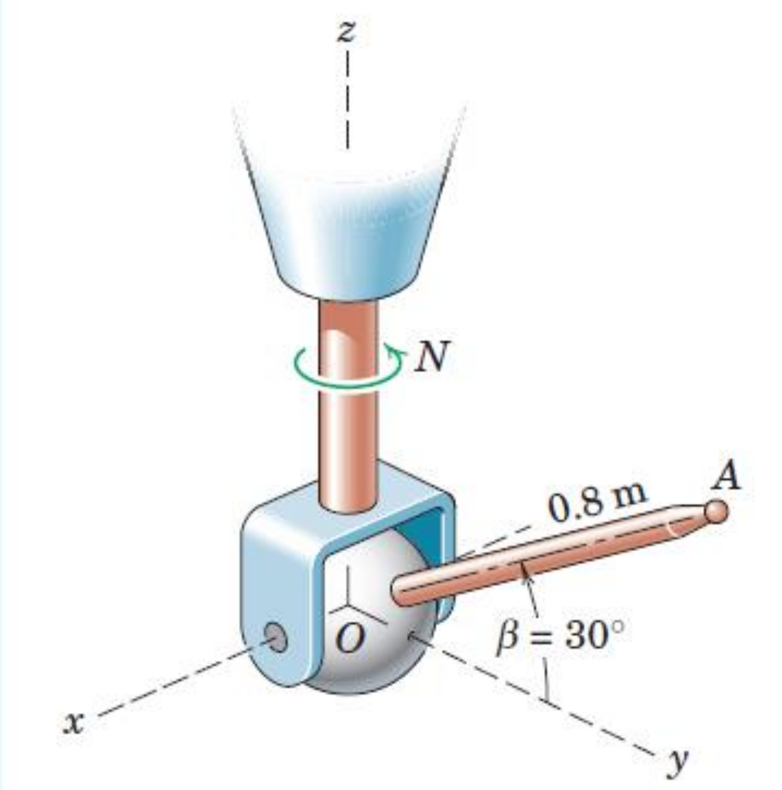
will have a component normal to $\boldsymbol{\omega}$ due to the change in direction of $\boldsymbol{\omega}$.

$$\vec{v} = \vec{\omega} \times \vec{r}$$

$$\vec{a} = \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times \left(\vec{\omega} \times \vec{r} \right)$$

Although any point on the rotation axis *n-n* momentarily will have zero velocity, it will not have zero acceleration as long as ω is changing its direction.

The 0.8-m arm OA for a remote-control mechanism is pivoted about the horizontal x-axis of the clevis, and the entire assembly rotates about the z-axis with a constant speed $N = 60$ rev/min. Simultaneously, the arm is being raised at the constant rate $\dot{\beta} = 4$ rad/s. For the position where $\beta = 30°$, determine (*a*) the angular velocity of OA, (*b*) the angular acceleration of OA, (*c*) the velocity of point A, and (*d*) the acceleration of point A. If, in addition to the motion described, the vertical shaft and point O had a linear motion, say, in the z-direction, would that motion change the angular velocity or angular acceleration of OA?



Solution. (a) Since the arm OA is rotating about both the x- and the z-axes, it has the components $\omega_x = \dot{\beta} = 4$ rad/s and $\omega_z = 2\pi N/60 = 2\pi(60)/60 = 6.28$ rad/s. The angular velocity is

$$\boldsymbol{\omega} = \boldsymbol{\omega}_x + \boldsymbol{\omega}_z = 4\mathbf{i} + 6.28\mathbf{k} \text{ rad/s} \qquad \textit{Ans.}$$

(b) The angular acceleration of OA is

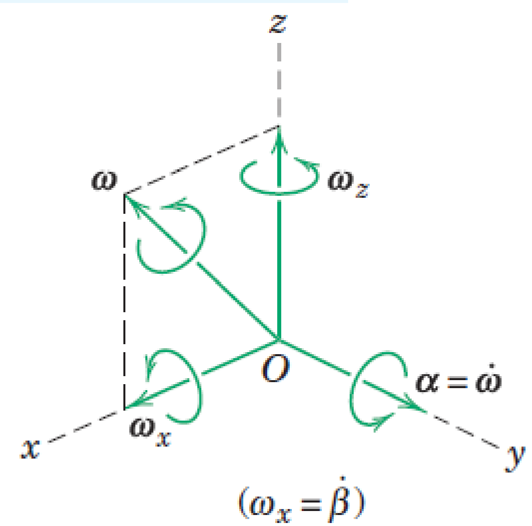
$$\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}} = \dot{\boldsymbol{\omega}}_x + \dot{\boldsymbol{\omega}}_z$$

Since $\boldsymbol{\omega}_z$ is not changing in magnitude or direction, $\dot{\boldsymbol{\omega}}_z = \mathbf{0}$. But $\boldsymbol{\omega}_x$ is changing direction and thus has a derivative which, from Eq. 7/3, is

$$\dot{\boldsymbol{\omega}}_x = \boldsymbol{\omega}_z \times \boldsymbol{\omega}_x = 6.28\mathbf{k} \times 4\mathbf{i} = 25.1\mathbf{j} \text{ rad/s}^2$$

Therefore,

$$\boldsymbol{\alpha} = 25.1\mathbf{j} + \mathbf{0} = 25.1\mathbf{j} \text{ rad/s}^2$$



(c) With the position vector of A given by $\mathbf{r} = 0.693\mathbf{j} + 0.4\mathbf{k}$ m, the velocity of A from Eq. 7/1 becomes

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 0 & 6.28 \\ 0 & 0.693 & 0.4 \end{vmatrix} = -4.35\mathbf{i} - 1.60\mathbf{j} + 2.77\mathbf{k} \text{ m/s} \qquad \textit{Ans.}$$

(d) The acceleration of A from Eq. 7/2 is

$$\mathbf{a} = \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

$$= \boldsymbol{\alpha} \times \mathbf{r} + \boldsymbol{\omega} \times \mathbf{v}$$

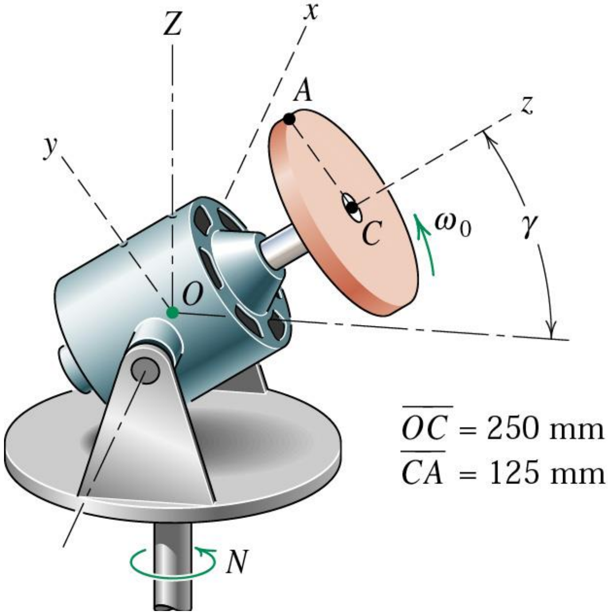
$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 25.1 & 0 \\ 0 & 0.693 & 0.4 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 0 & 6.28 \\ -4.35 & -1.60 & 2.77 \end{vmatrix}$$

$$= (10.05\mathbf{i}) + (10.05\mathbf{i} - 38.4\mathbf{j} - 6.40\mathbf{k})$$

2 $$= 20.1\mathbf{i} - 38.4\mathbf{j} - 6.40\mathbf{k} \text{ m/s}^2 \qquad \textit{Ans.}$$

The angular motion of OA depends only on the angular changes N and $\dot{\beta}$, so any linear motion of O does not affect $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$.

Example

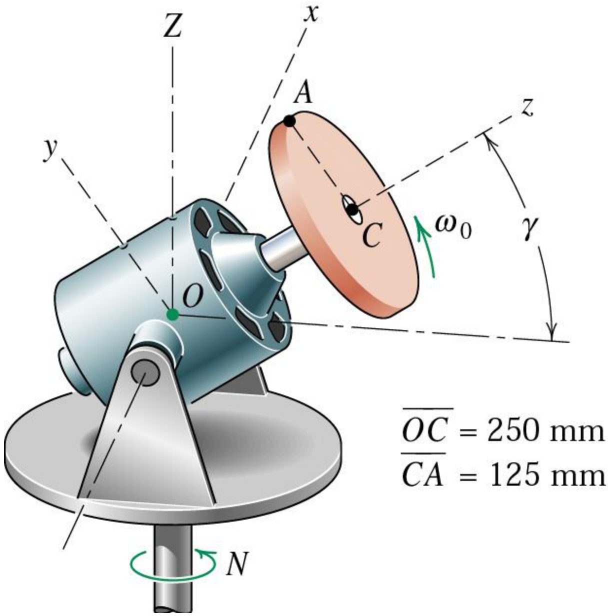


The electric motor with an attached disk is running at a constant low speed of 120 rev/min in the direction shown. Its housing and mounting base are initially rest. The entire assembly is next set in rotation about the vertical Z-axis at the constant rate N=60 rev/min with a fixed angle γ of 30º. Determine **(a)** the angular velocity and angular acceleration of the disk, **(b)** the space and body cone, and **(c)** the velocity and acceleration of point A at the top of the disk for the instant shown.

The axes x-y-z with unit vectors $\vec{i}$, $\vec{j}$, $\vec{k}$ are attached to the motor frame.

$$\vec{K} = \vec{j}\cos\gamma + \vec{k}\sin\gamma$$

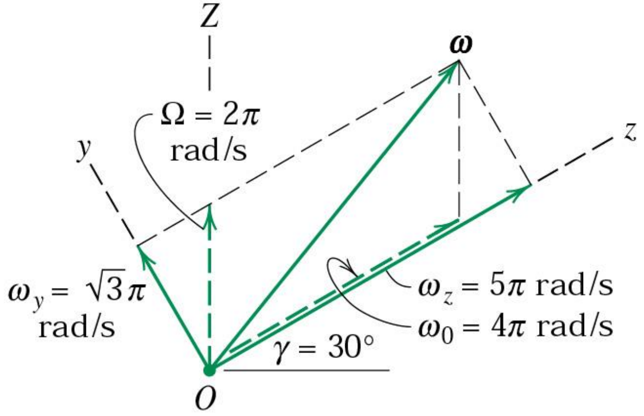
(a) the angular velocity and angular acceleration of the disk



The rotor and disk have two components of angular velocity : ω_o=120(2π)/60=4π rad/s about the z-axis and Ω= 60(2π)/60=2π rad/s about the Z-axis.

$$\vec{\omega} = \vec{\omega}_o + \vec{\Omega} = \omega_o \vec{k} + \Omega \vec{K}$$

$$\vec{\omega} = \omega_o \vec{k} + \Omega\left(\vec{j}\cos\gamma + \vec{k}\sin\gamma\right) = \pi\left(\sqrt{3}\,\vec{j} + 5\vec{k}\right)$$



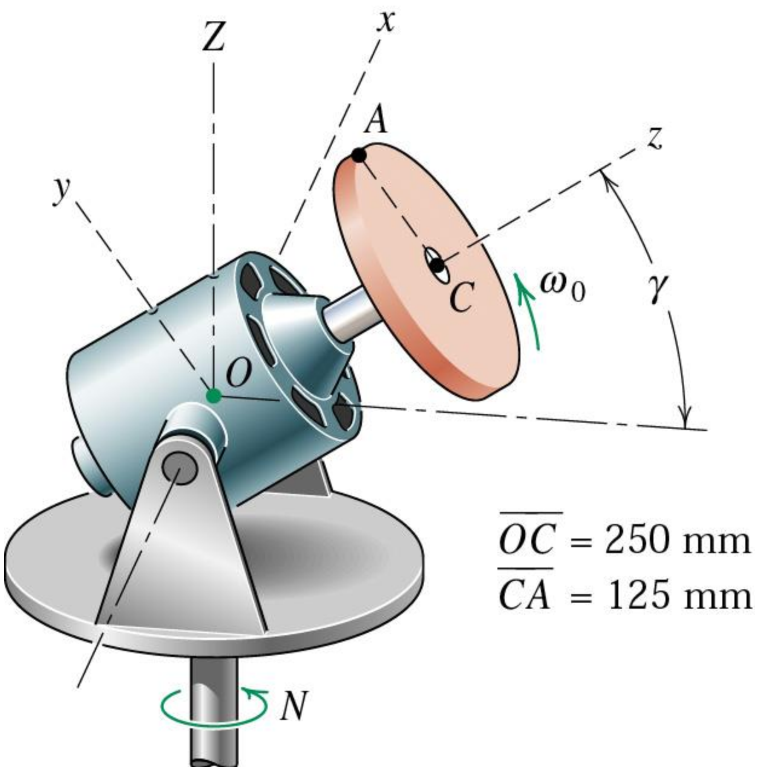
The angular acceleration of the disk

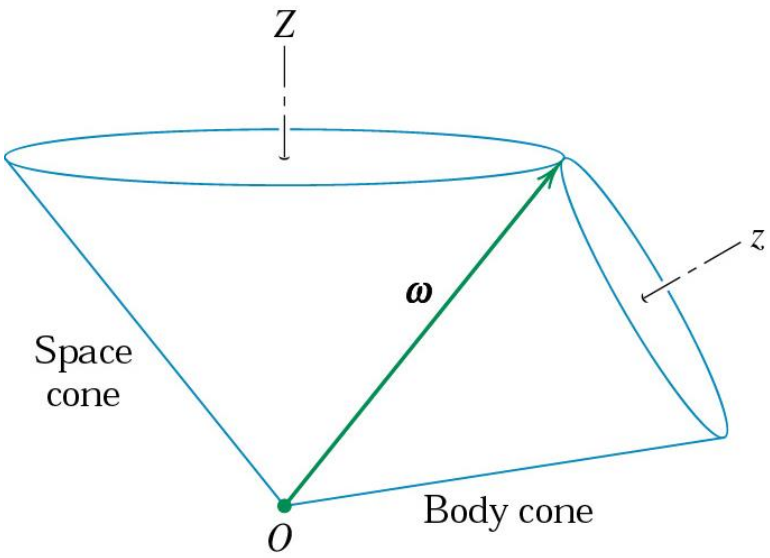
$$\vec{\alpha} = \dot{\vec{\omega}} = \vec{\Omega} \times \vec{\omega}$$

$$\vec{\alpha} = \Omega\left(\vec{j}\cos\gamma + \vec{k}\sin\gamma\right) \times \left[(\Omega\cos\gamma)\vec{j} + (\omega_o + \Omega\sin\gamma)\vec{k}\right]$$

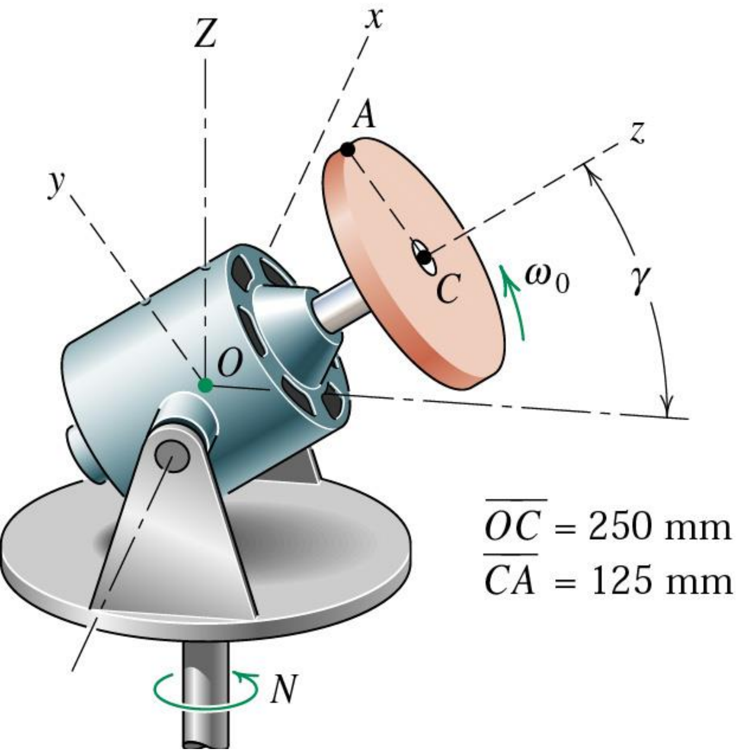
$$\vec{\alpha} = \Omega\omega_o\cos\gamma\,\vec{i} = 68.4\vec{i}$$

b) the space and body cone





(c) the velocity and acceleration of point A at the top of the disk



$$\vec{r}_{A/O} = 0.125\vec{j} + 0.25\vec{k}$$

$$\vec{v} = \vec{\omega} \times \vec{r}$$

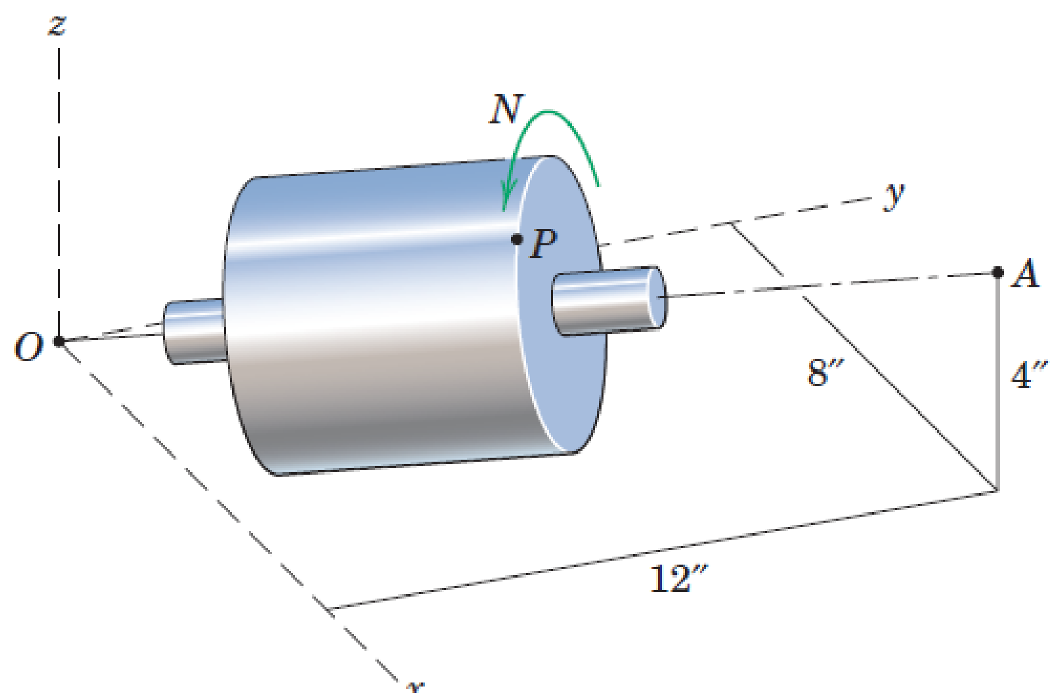
$$\vec{v} = \pi\left(\sqrt{3}\vec{j} + 5\vec{k}\right) \times \left(0.125\vec{j} + 0.25\vec{k}\right)$$

$$\vec{v} = -0.192\pi\vec{i}$$

$$\vec{a} = \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times \left(\vec{\omega} \times \vec{r}\right) = \vec{\alpha} \times \vec{r} + \vec{\omega} \times \vec{v}$$

$$\vec{a} = -26.6\vec{j} + 11.83\vec{k}$$

7/3 The solid cylinder is rotating about the fixed axis OA with a constant speed $N = 600$ rev/min in the direction shown. If the x- and y-components of the velocity of point P are 12 ft/sec and -6 ft/sec, determine its z-component of velocity and the radial distance R from P to the rotation axis. Also find the magnitude of the acceleration of P.



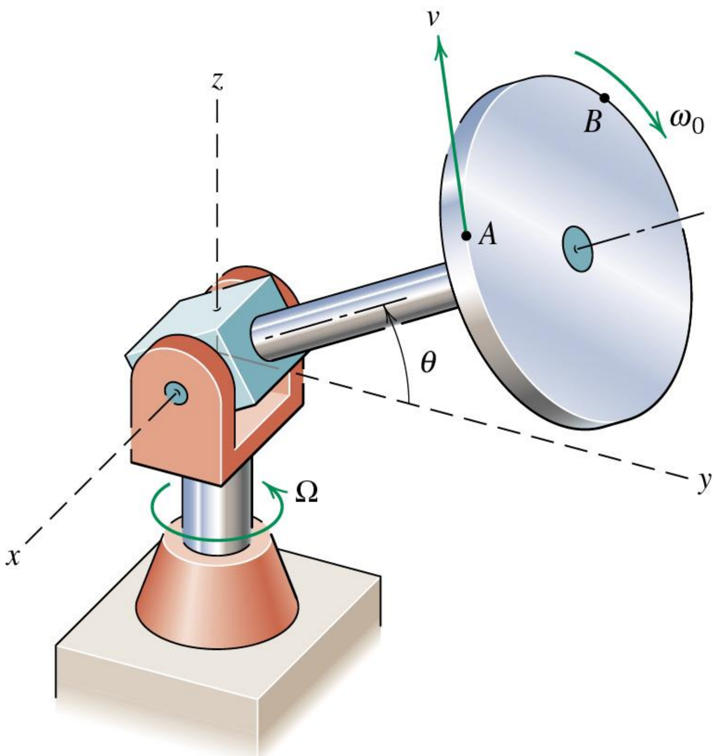
The rotor and shaft are mounted in a clevis which can rotate about the z-axis with an angular Ω. With Ω=0 and θ constant, the rotor has an angular velocity

$$\vec{\omega}_o = -4\vec{j} - 3\vec{k} \quad \text{rad/s.}$$

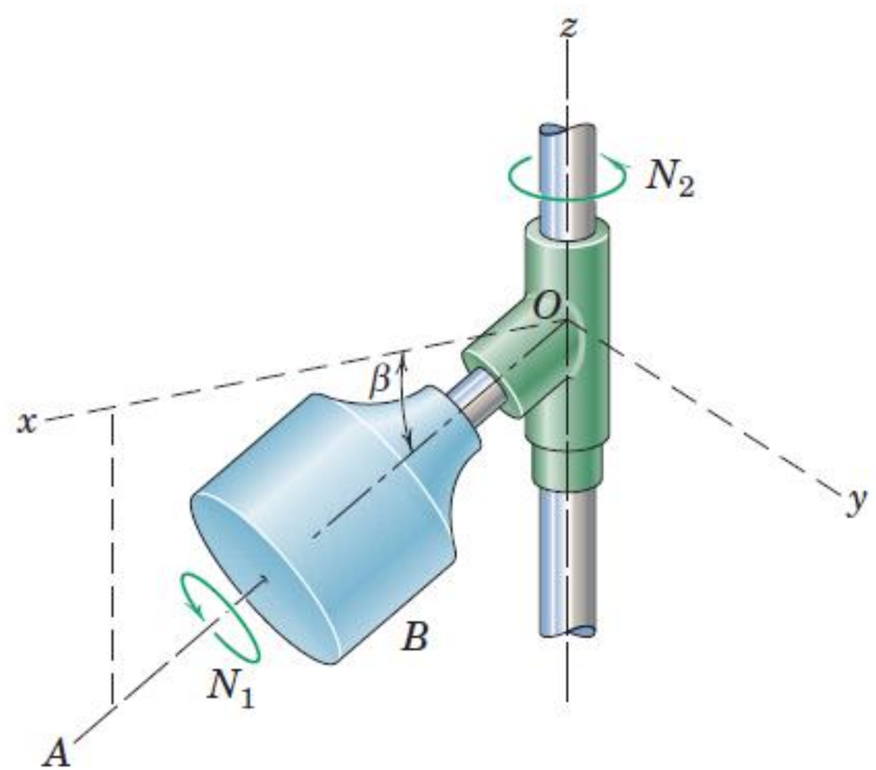
Find the velocity v_A of point A on the rim if its position vector at the instant.

$$\vec{r} = 0.5\vec{i} + 1.2\vec{j} + 1.1\vec{k}$$

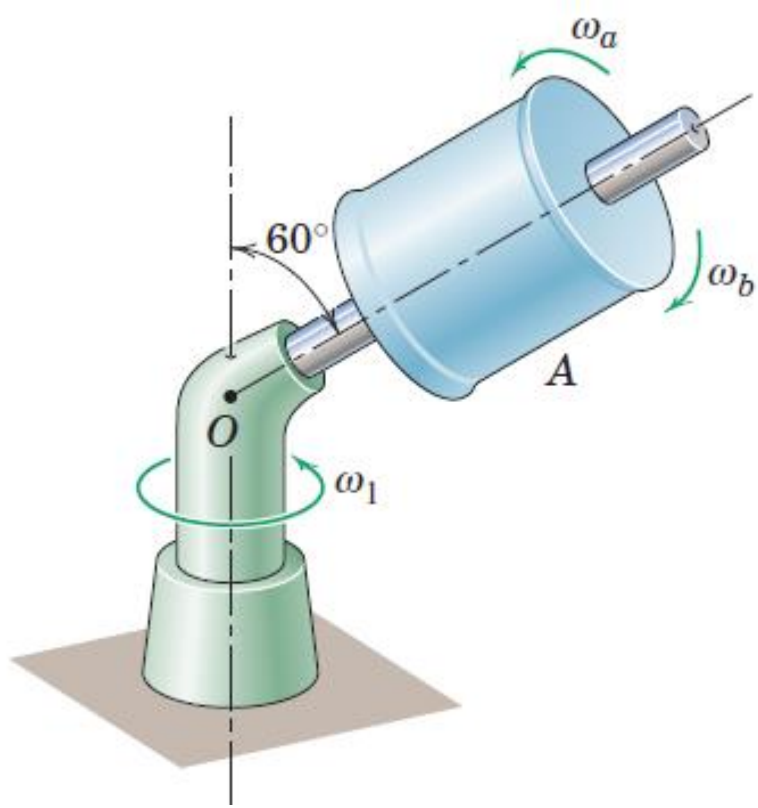
What is the rim speed v_B of any point B?

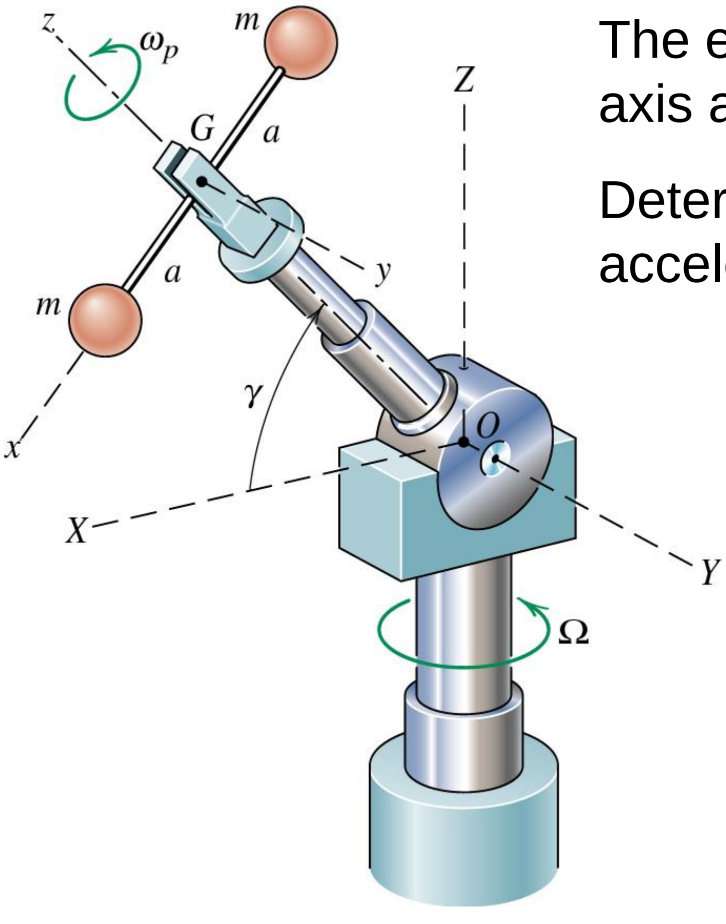


7/7 The rotor B spins about its inclined axis OA at the speed $N_1 = 200$ rev/min, where $\beta = 30°$. Simultaneously, the assembly rotates about the vertical z-axis at the rate N_2. If the total angular velocity of the rotor has a magnitude of 40 rad/s, determine N_2.



7/13 The spool A rotates about its axis with an angular velocity of 20 rad/s, first in the sense of ω_a and second in the sense of ω_b. Simultaneously, the assembly rotates about the vertical axis with an angular velocity $\omega_1 = 10$ rad/s. Determine the magnitude ω of the total angular velocity of the spool and construct the body and space cones for the spool for each case.





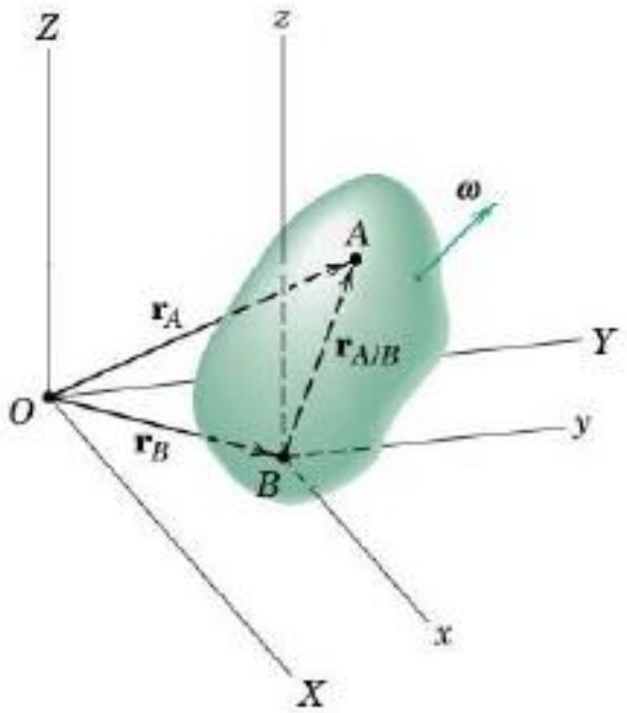
In manipulating the dumbbell, the jaws of robotic device have an angular velocity ω_P=2 rad/s about the axis OG with γ fixed at 60º.

The entire assembly rotates about the vertical Z-axis at the constant rate Ω=0.8 rad/s.

Determine the angular velocity and angular acceleration of the dumbbell.

General motion

Translating Reference Axes



The velocity v and acceleration a of any other point A in the body are given by

$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}$$

$$\mathbf{a}_A = \mathbf{a}_B + \mathbf{a}_{A/B}$$

In rigid body, general motion as a translation of the body with the motion B plus a rotation of the body about B

$$\mathbf{v}_A = \mathbf{v}_B + \boldsymbol{\omega} \times \mathbf{r}_{A/B}$$

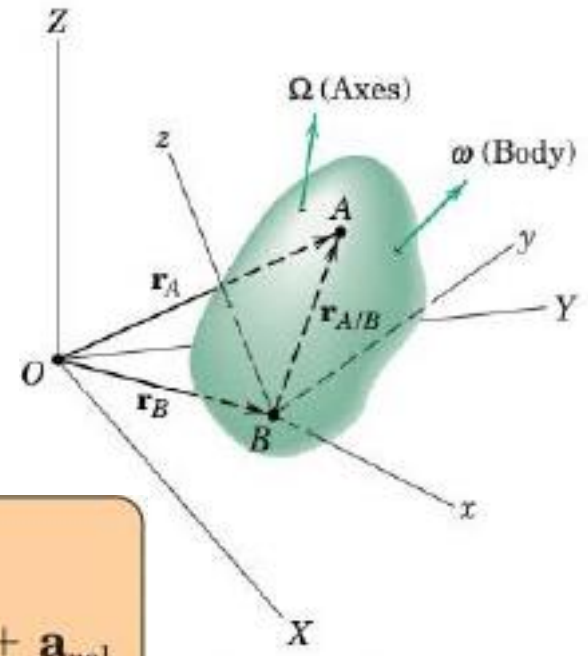
$$\mathbf{a}_A = \mathbf{a}_B + \dot{\boldsymbol{\omega}} \times \mathbf{r}_{A/B} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{A/B})$$

Rotating Reference Axes

Time derivatives of the rotating unit vectors attached to x-y-z

$$\dot{\mathbf{i}} = \boldsymbol{\Omega} \times \mathbf{i} \qquad \dot{\mathbf{j}} = \boldsymbol{\Omega} \times \mathbf{j} \qquad \dot{\mathbf{k}} = \boldsymbol{\Omega} \times \mathbf{k}$$

The expression for the velocity and acceleration of point A



$$\mathbf{v}_A = \mathbf{v}_B + \boldsymbol{\Omega} \times \mathbf{r}_{A/B} + \mathbf{v}_{\text{rel}}$$

$$\mathbf{a}_A = \mathbf{a}_B + \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{A/B} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{A/B}) + 2\boldsymbol{\Omega} \times \mathbf{v}_{\text{rel}} + \mathbf{a}_{\text{rel}}$$

where $\mathbf{v}_{\text{rel}} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k}$ and $\mathbf{a}_{\text{rel}} = \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j} + \ddot{z}\mathbf{k}$ are, respectively, the velocity and acceleration of point A measured relative to x-y-z by an observer attached to x-v-z.

The time derivative of a vector V as measured in the fixed X-Y system and time derivative of V as measured relative to the rotating x-y system.

For 3D case

$$\left(\frac{d\mathbf{V}}{dt}\right)_{XYZ} = \left(\frac{d\mathbf{V}}{dt}\right)_{xyz} + \mathbf{\Omega} \times \mathbf{V}$$

Applying this transformation to the relative-position vector

$$\left(\frac{d\mathbf{r}_A}{dt}\right)_{XYZ} = \left(\frac{d\mathbf{r}_B}{dt}\right)_{XYZ} + \left(\frac{d\mathbf{r}_{A/B}}{dt}\right)_{xyz} + \mathbf{\Omega} \times \mathbf{r}_{A/B}$$

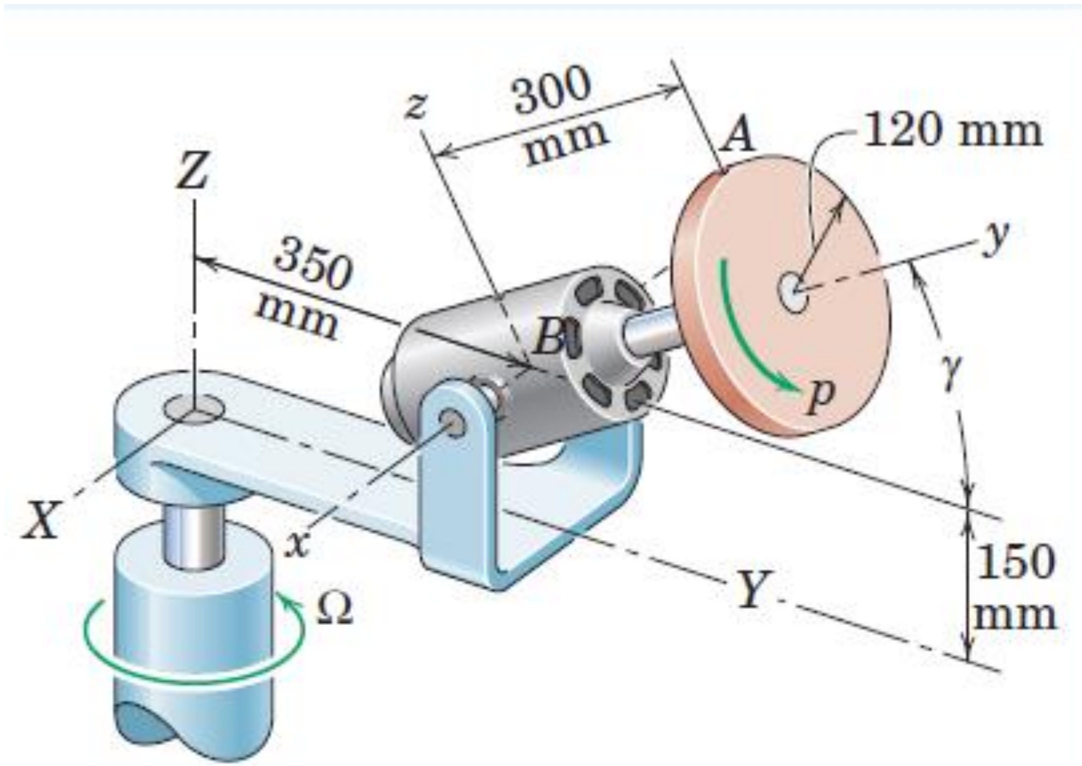
First time derivatives

$$\left(\frac{d[\quad]}{dt}\right)_{XYZ} = \left(\frac{d[\quad]}{dt}\right)_{xyz} + \mathbf{\Omega} \times [\quad]$$

Second time derivatives

$$\left(\frac{d^2[\quad]}{dt^2}\right)_{XYZ} = \left(\frac{d^2[\quad]}{dt^2}\right)_{xyz} + \dot{\mathbf{\Omega}} \times [\quad] + \mathbf{\Omega} \times (\mathbf{\Omega} \times [\quad]) + 2\mathbf{\Omega} \times \left(\frac{d[\quad]}{dt}\right)_{xyz}$$

The motor housing and its bracket rotate about the Z-axis at the constant rate $\Omega = 3$ rad/s. The motor shaft and disk have a constant angular velocity of spin $p = 8$ rad/s with respect to the motor housing in the direction shown. If γ is constant at 30°, determine the velocity and acceleration of point A at the top of the disk and the angular acceleration $\boldsymbol{\alpha}$ of the disk.



1 **Solution.** The rotating reference axes *x-y-z* are attached to the motor housing, and the rotating base for the motor has the momentary orientation shown with respect to the fixed axes *X-Y-Z*. We will use both *X-Y-Z* components with unit vectors **I**, **J**, **K** and *x-y-z* components with unit vectors **i**, **j**, **k**. The angular velocity of the *x-y-z* axes becomes $\mathbf{\Omega} = \Omega\mathbf{K} = 3\mathbf{K}$ rad/s.

Velocity. The velocity of *A* is given by the first of Eqs. 7/6

$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{\Omega} \times \mathbf{r}_{A/B} + \mathbf{v}_{\text{rel}}$$

where

$$\mathbf{v}_B = \mathbf{\Omega} \times \mathbf{r}_B = 3\mathbf{K} \times 0.350\mathbf{J} = -1.05\mathbf{I} = -1.05\mathbf{i} \text{ m/s}$$

2

$$\mathbf{\Omega} \times \mathbf{r}_{A/B} = 3\mathbf{K} \times (0.300\mathbf{j} + 0.120\mathbf{k})$$

$$= (-0.9 \cos 30°)\mathbf{i} + (0.36 \sin 30°)\mathbf{i} = -0.599\mathbf{i} \text{ m/s}$$

$$\mathbf{v}_{\text{rel}} = \mathbf{p} \times \mathbf{r}_{A/B} = 8\mathbf{j} \times (0.300\mathbf{j} + 0.120\mathbf{k}) = 0.960\mathbf{i} \text{ m/s}$$

Thus,

$$\mathbf{v}_A = -1.05\mathbf{i} - 0.599\mathbf{i} + 0.960\mathbf{i} = -0.689\mathbf{i} \text{ m/s}$$ *Ans.*

Acceleration. The acceleration of A is given by the second of Eqs. 7/6

$$\mathbf{a}_A = \mathbf{a}_B + \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{A/B} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{A/B}) + 2\boldsymbol{\Omega} \times \mathbf{v}_{\text{rel}} + \mathbf{a}_{\text{rel}}$$

where

$$\mathbf{a}_B = \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_B) = 3\mathbf{K} \times (3\mathbf{K} \times 0.350\mathbf{J}) = -3.15\mathbf{J}$$

$$= 3.15(-\mathbf{j} \cos 30° + \mathbf{k} \sin 30°) = -2.73\mathbf{j} + 1.575\mathbf{k} \text{ m/s}^2$$

$$\dot{\boldsymbol{\Omega}} = \mathbf{0}$$

2 $$\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{A/B}) = 3\mathbf{K} \times [3\mathbf{K} \times (0.300\mathbf{j} + 0.120\mathbf{k})]$$

$$= 3\mathbf{K} \times (-0.599\mathbf{i}) = -1.557\mathbf{j} + 0.899\mathbf{k} \text{ m/s}^2$$

$$2\boldsymbol{\Omega} \times \mathbf{v}_{\text{rel}} = 2(3\mathbf{K}) \times 0.960\mathbf{i} = 5.76\mathbf{J}$$

$$= 5.76(\mathbf{j} \cos 30° - \mathbf{k} \sin 30°) = 4.99\mathbf{j} - 2.88\mathbf{k} \text{ m/s}^2$$

$$\mathbf{a}_{\text{rel}} = \mathbf{p} \times (\mathbf{p} \times \mathbf{r}_{A/B}) = 8\mathbf{j} \times [8\mathbf{j} \times (0.300\mathbf{j} + 0.120\mathbf{k})]$$

$$= -7.68\mathbf{k} \text{ m/s}^2$$

Substituting into the expression for $\mathbf{a}_A$ and collecting terms give us

$$\mathbf{a}_A = 0.703\mathbf{j} - 8.09\mathbf{k} \text{ m/s}^2$$

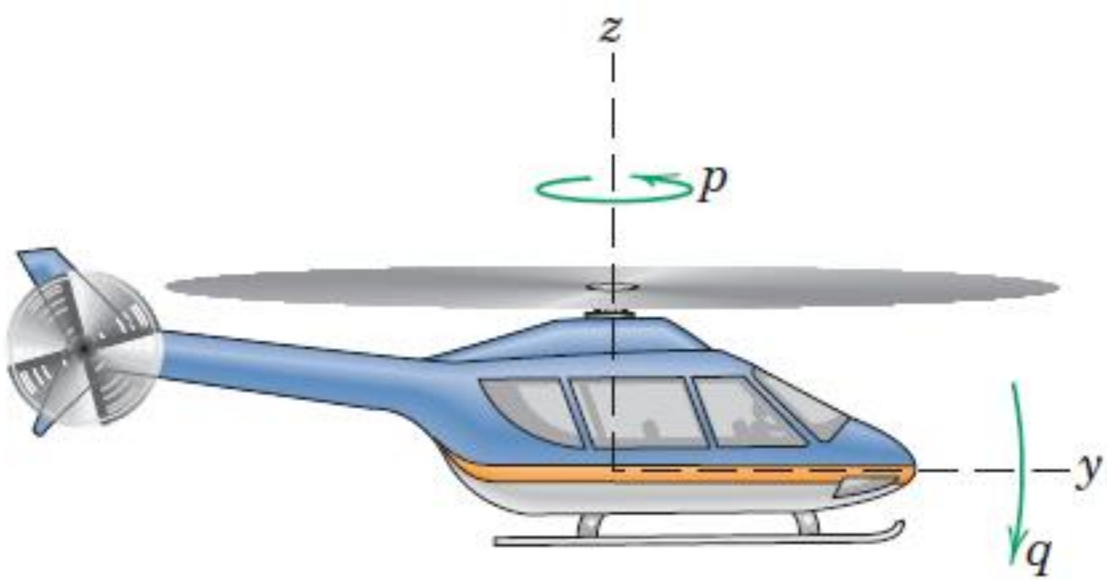
and $$a_A = \sqrt{(0.703)^2 + (8.09)^2} = 8.12 \text{ m/s}^2$$

Angular Acceleration. Since the precession is steady, we may use Eq. 7/3 to give us

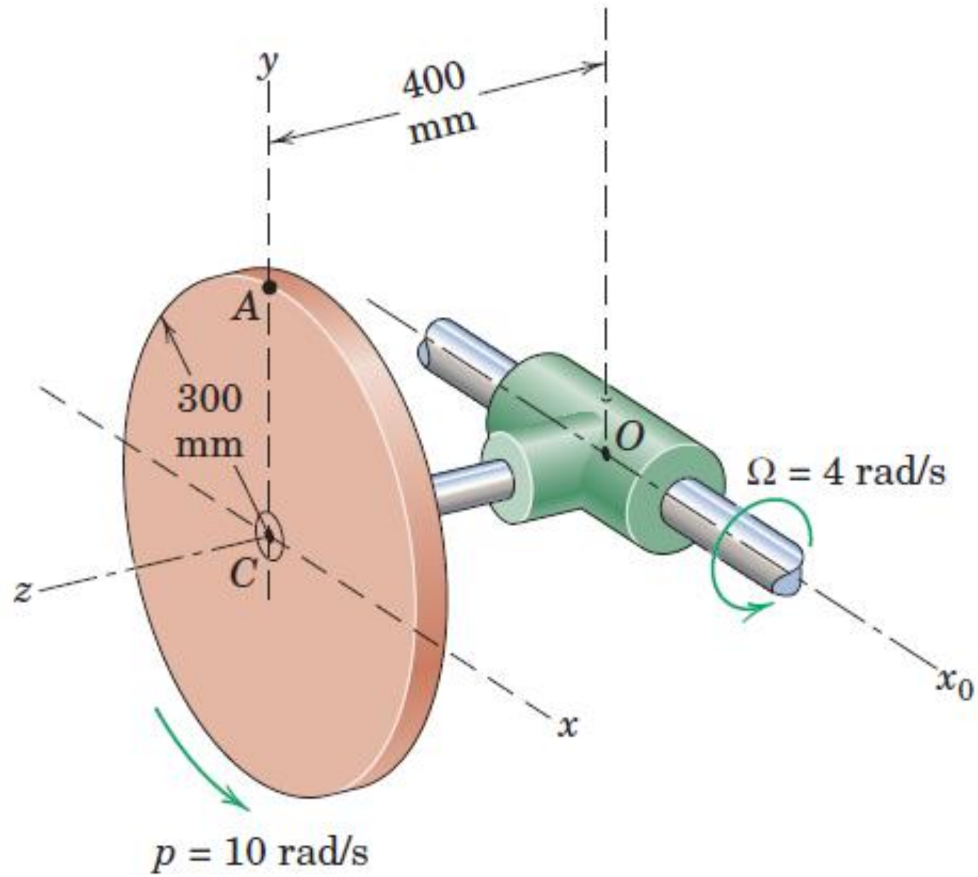
$$\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}} = \boldsymbol{\Omega} \times \boldsymbol{\omega} = 3\mathbf{K} \times (3\mathbf{K} + 8\mathbf{j})$$

$$= \mathbf{0} + (-24 \cos 30°)\mathbf{i} = -20.8\mathbf{i} \text{ rad/s}^2 \qquad \textit{Ans.}$$

7/30 The helicopter is nosing over at the constant rate q rad/s. If the rotor blades revolve at the constant speed p rad/s, write the expression for the angular acceleration $\boldsymbol{\alpha}$ of the rotor. Take the y-axis to be attached to the fuselage and pointing forward perpendicular to the rotor axis.

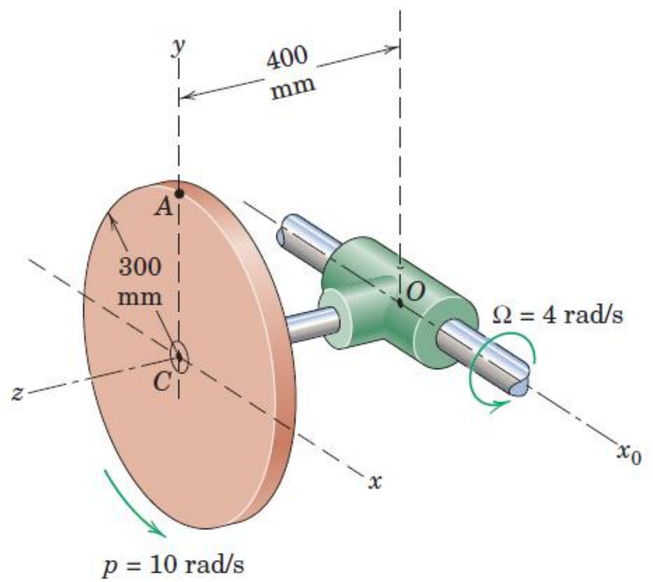


7/31 The collar at O and attached shaft OC rotate about the fixed x_0-axis at the constant rate $\Omega = 4$ rad/s. Simultaneously, the circular disk rotates about OC at the constant rate $p = 10$ rad/s. Determine the magnitude of the total angular velocity $\boldsymbol{\omega}$ of the disk and find its angular acceleration $\boldsymbol{\alpha}$.

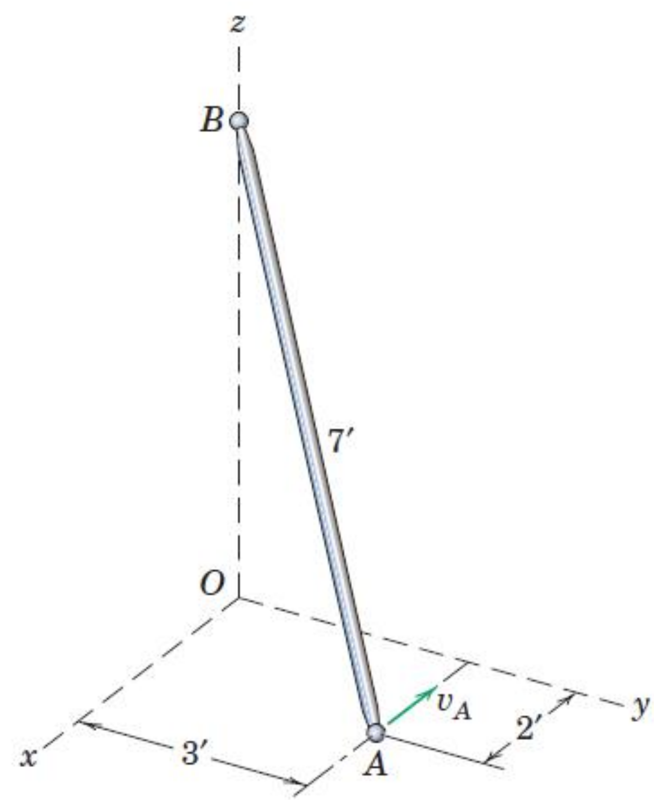


7/32 If the angular rate p of the disk in Prob. 7/31 is increasing at the rate of 6 rad/s per second and if Ω remains constant at 4 rad/s, determine the angular acceleration $\boldsymbol{\alpha}$ of the disk at the instant when p reaches 10 rad/s.

7/33 For the conditions of Prob. 7/31, determine the velocity $\mathbf{v}_A$ and acceleration $\mathbf{a}_A$ of point A on the disk as it passes the position shown. Reference axes x-y-z are attached to the collar at O and its shaft OC.



7/35 End A of the rigid link is confined to move in the $-x$-direction while end B is confined to move along the z-axis. Determine the component $\boldsymbol{\omega}_n$ normal to AB of the angular velocity of the link as it passes the position shown with $v_A = 3$ ft/sec.



7/43 The circular disk of 100-mm radius rotates about its z-axis at the constant speed $p = 240$ rev/min, and arm OCB rotates about the Y-axis at the constant speed $N = 30$ rev/min. Determine the velocity **v** and acceleration **a** of point A on the disk as it passes the position shown. Use reference axes x-y-z attached to the arm OCB.