Kinematics of Rigid bodies

1 Multireference analysis.

- To analyze complicated motion in a more simple systematic way by using several reference.
- Motion of a particle is often known relative to a moving body, to which we can fix ~~fix~~ a reference xyz, while the motion of the plane is known relative ~~form~~ to an inertial refrence XYZ (such as ground).

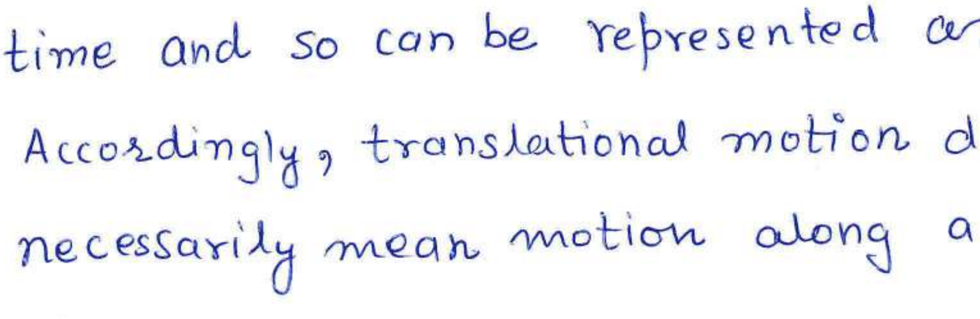
→ Since Newton's Law in the form $F = ma$, is valid only for an inertial reference. Hence to use Newton's Law for the particle we must express the acceleration of the particle relative to the inertial reference directly.

Rigid Body

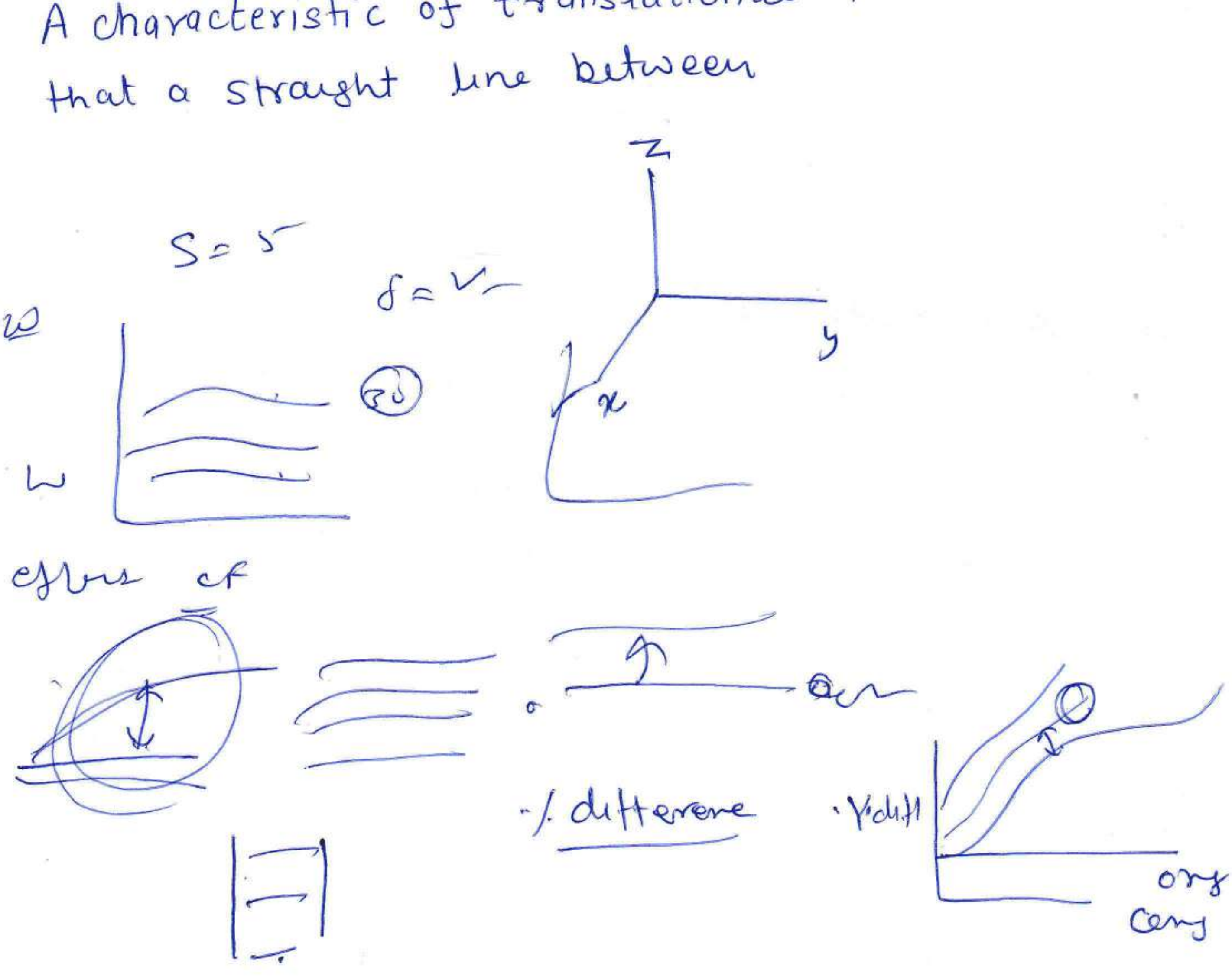
A rigid body is considered to be composed of a continuous distribution of particles having fixed distances between each other

Translation: if a body moves so that all the particles have at time t the same velocity relative to some reference, the body is said to be in translation relative to this reference at this time.

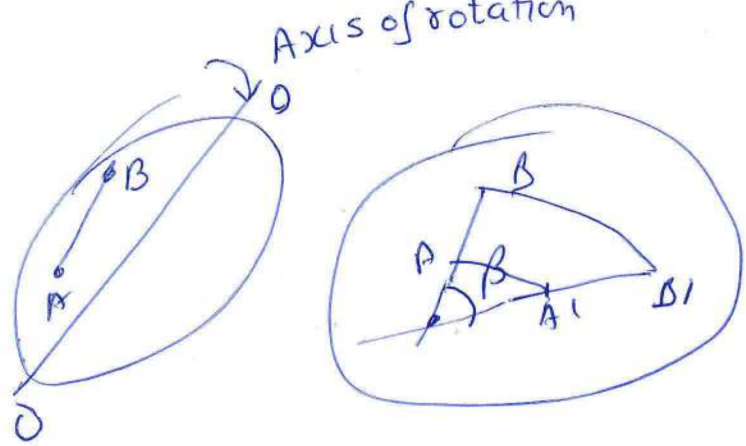
Velocity of a translating body can vary with time and so can be represented as V(t).
Accordingly, translational motion does not necessarily mean motion along a straight line.
A characteristic of translational motion is that a straight line between



effors of



Rotation: if a rigid body moves so that along some straight line all the particles of the body have zero velocity relative to some reference, the body is said to be in rotation relative to this reference. The line of stationary particles is called the axis of rotation.



Finite Rotation: ~~are~~ have magnitude and a direction along the axis of rotation, are not vectors. Superposition of rotations is not commutative and therefore rotations do not add according to the parallelogram law, which, ~~you~~ is a requirement of all vector quantities.

Infinitesimal rotation - ~~stat~~ satisfy in the limit the commutative law of addition, so that infinitesimal rotations $d\beta$ are vector quantities.

Angular Velocity → Vector quantity

magnitude - $d\beta/dt$

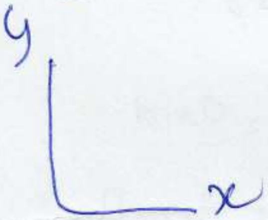
Direction → parallel to the axis of rotation.

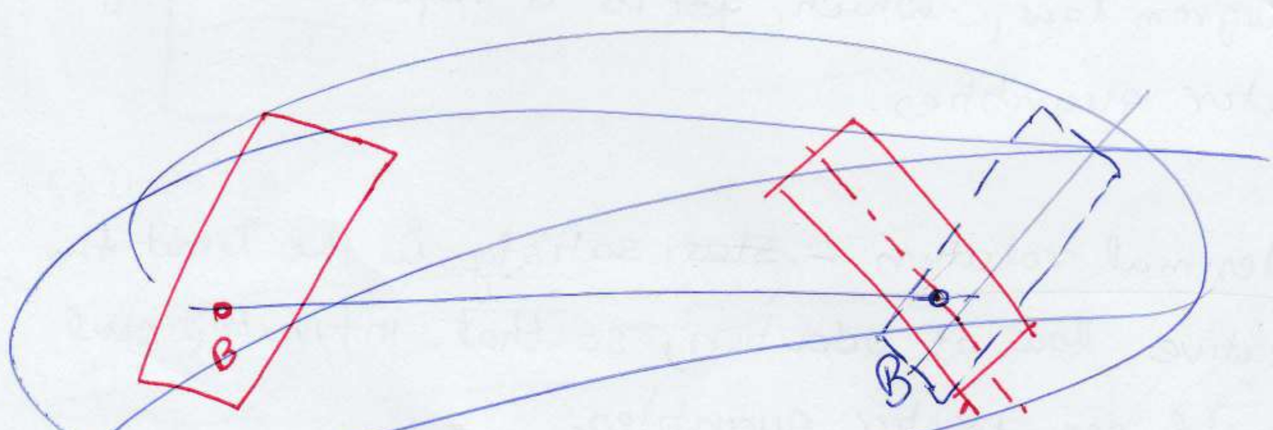
Sense is According to the right hand Screw rule.

ω = Angular velocity

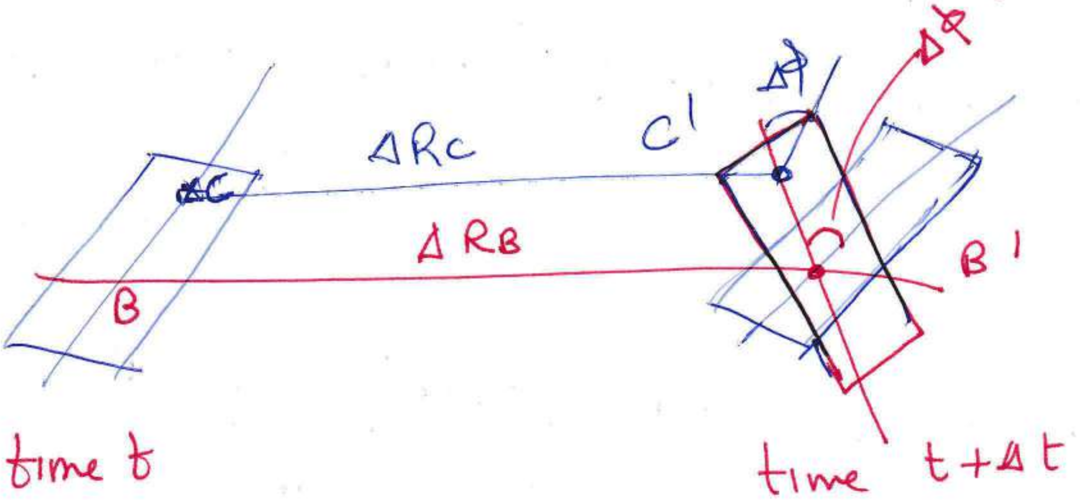
General Motion of Rigid body

= Translation motion + Rotational Motion.





Translation & Rotation of a rigid body



if we choose some other point C

Displacements ΔR_B & ΔR_C different

But $\Delta\phi$ - rotation → no difference.

In General, ΔR and the axis of rotation will depend on the point chosen, while the amount of rotation $\Delta\phi$ will be the same for all such points.

$\frac{\Delta R}{\Delta t}$ – Average translation velocity

$\frac{\Delta \phi}{\Delta t}$ – average rotational speed

if $\Delta t \to 0$

Then Instantaneous translation

ω – Instantaneous rotation (angular velocity)

CHASLES' Theorem.

1. ~~Sle~~ select any point B in the body. Assume that all the particles of the body have at the time 't' a velocity equal to V_B, the actual velocity of point B.

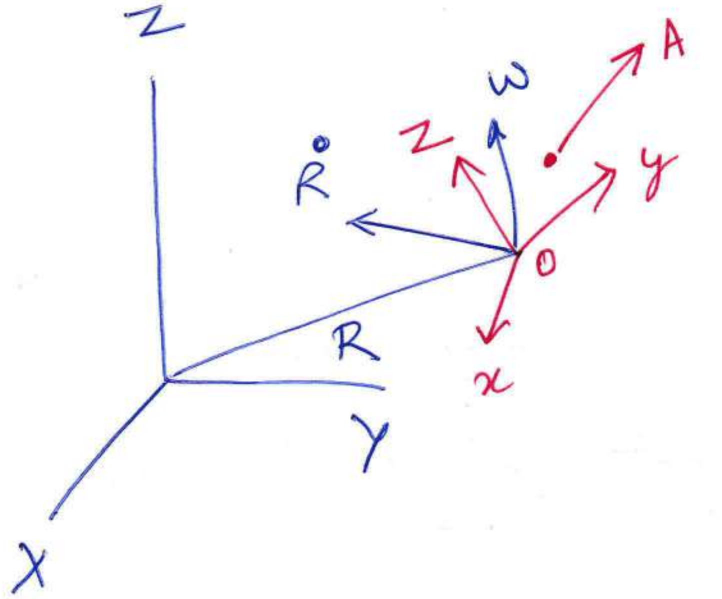
2. Superpose a pure rotational velocity ~~ve~~ about an axis of rotation going through point B.

with V_B & ω – The actual instantaneous motion of the body is determined, & ω will be the same for all points B which might be choses.

<u>Note</u>: Translational veocities & axis of rotation changes when different points B are chosen.

Actual Instaneous axis of Rotation at time t is the one going through those points of the body having zero velocity at time t.

Derivative of a Vector fixed in a Moving Reference.



Two References XYZ & xyz move arbitrarily relative to each other.

Assume we are observing xyz from XYZ, since, a reference is a rigid system, we can apply Chasles' theorem to reference xyz.

→ Choose a origin - O

→ Superpose a Translational Velocity $\dot{R}$ equal to the Velocity of O

→ rotational Velocity ω with an axis of rotation through O.

Assume

$\overline{A}$ = Vector of fixed length

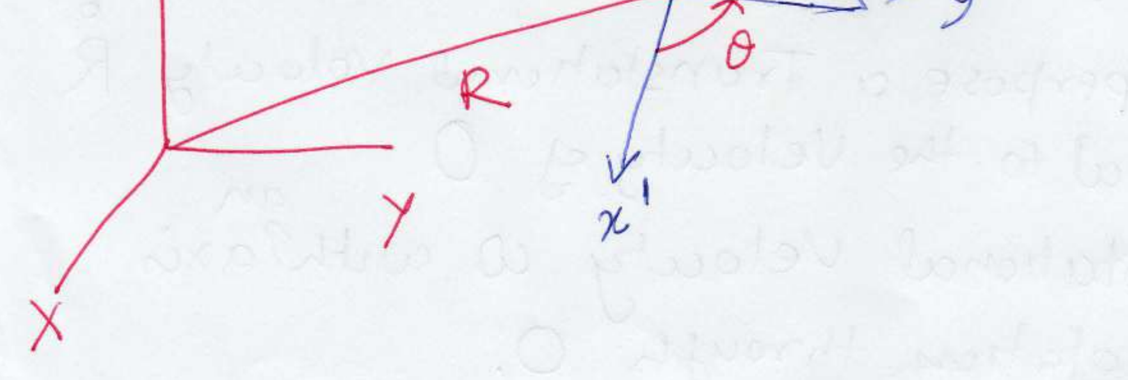
fixed orientation as seen from xyz

$$\therefore \left(\frac{d\vec{A}}{dt}\right)_{xyz} = 0$$

However, as seen from XYZ, the time rate of chang A will not necessarily be zero.

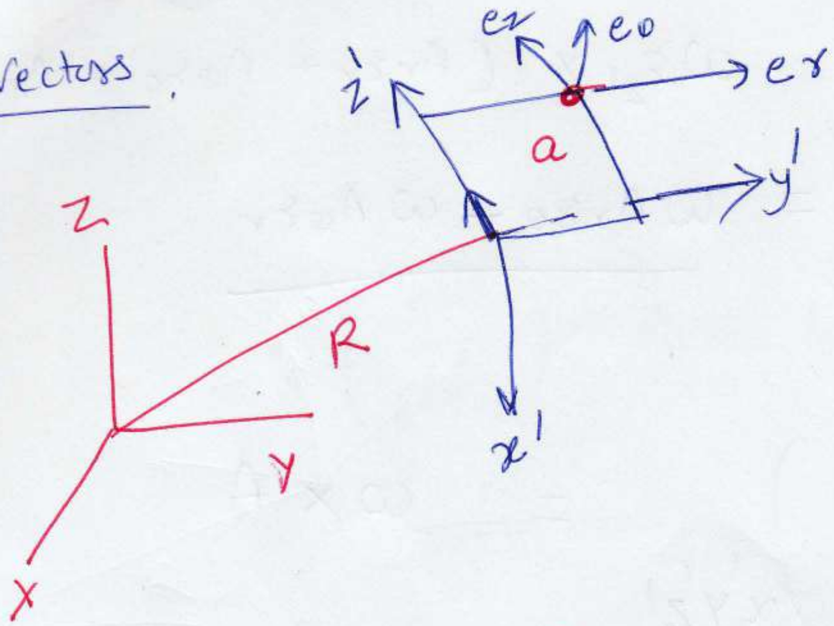
To Evalute $\left(\frac{d\vec{A}}{dt}\right)_{XYZ}$

To Best observe this rotation, we shall employ at O a stationary reference $x', y'z'$ positioned so that z' coincides with the axis of rotation.



Cylindrical Components A_r, A_θ, & A_z.

Unit Vectors.



unit vectors e_r, e_θ, e_z

We can express $\underline{\underline{A}}$

$$A = A_r e_r + A_\theta e_\theta + A_z \bar{e}_z$$

as A rotates about z', the values of the cylindrical scalar components of A for $x', y' z'$, namely A_r, A_θ & A_z, do not change.

$$\mathring{A}_r = \mathring{A}_\theta = \mathring{A}_z = 0 \quad \text{and} \quad \dot{e}_z = 0$$

$$\left(\frac{dA}{dt}\right)_{x'y'z'} = A_r \left(\frac{de_r}{dt}\right)_{x'y'z'} + A_\theta \left(\frac{de_\theta}{dt}\right)_{x'y'z'}$$

$$\left(\frac{dA}{dt}\right)_{x'y'z'} = \underline{A_r \omega e_\theta - A_\theta \omega e_r}$$

Now

$$\omega \times A = \omega \varepsilon_{z} \times (A_r \varepsilon_r + A_\theta \varepsilon_\theta + A_z \varepsilon_z)$$

$$= \omega A_r \varepsilon_\theta - \omega A_\theta \varepsilon_r$$

Now

$$\left(\frac{dA}{dt}\right)_{x'y'z'} = \omega \times A$$

Since $x'y'z'$ is stationary relative to XYZ, we would observe the same time derivative from the latter reference.

$$\left(\frac{dA}{dt}\right)_{XYZ} = \omega \times A$$

$\underline{1}$ $\left(\frac{dA}{dt}\right)_{XYZ}$ — depends only on the vectors ω & A and not on their line of actions.

Time rate of change of A fixed in xyz is not altered when

- Vector A is fixed at some other location in xyz provided the vector itself is not changed.
- Actual axis of rotation of xyz is shifted to new parallel position.

$$\left(\frac{dA}{dt}\right)_{xyz} = \omega \times A$$

$$\left(\frac{d^2A}{dt^2}\right) = \left(\frac{d\omega}{dt}\right)_{xyz} \times A + \omega \times \left(\frac{dA}{dt}\right)_{xyz}$$

$$= \overset{\circ}{\omega} \times A + \omega \times (\omega \times A)$$

How to evaluate triple cross product

$$\omega_1 \hat{k} \times (\omega_1 \hat{k} \times C\hat{j}) = -\omega_1^2 C\hat{j}$$

$\Rightarrow$ The product is minus the product of the scalars and has a direction corresponding to the last unit vector $\hat{j}$

Example: Angular velocity of body A relative to body B is given ω_1, while the angular velocity of body B relative to ground is ω_2.

- Total angular velocity ω_T of body A relative to ground: $\boxed{\omega_1 + \omega_2 = \omega_T}$

$$\boxed{\omega_1 = \omega_T - \omega_2}$$

Example

$$\dot{\omega} = \underline{\dot{\omega}_1} + \dot{\omega}_2$$

$\downarrow$

$$\left(\frac{d\omega_2}{dt}\right)_{xyz} = \frac{dj}{dt}\hat{k}$$

$$\frac{d\omega}{dt}j + \omega\frac{dj}{dt}$$

$\omega_2(\underline{\omega_1 \times \hat{i}})$ $\quad \left.\frac{dj}{dt}\right|_{xyz}$

$\frac{dj}{dt}$

$\underline{\omega_1}\,\hat{i}$

$$\frac{d\omega}{dt} + \omega_1\frac{di}{dt}$$

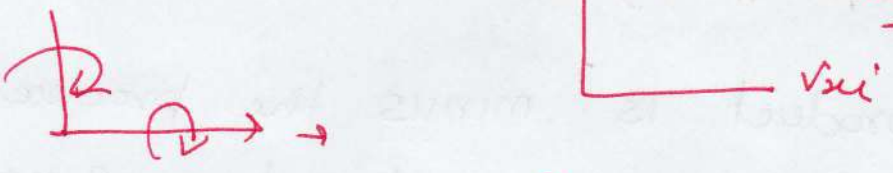
$$\omega_2(\omega_1 \cancel{k} + \omega_2 \cancel{j}) \times \hat{k}$$

ω

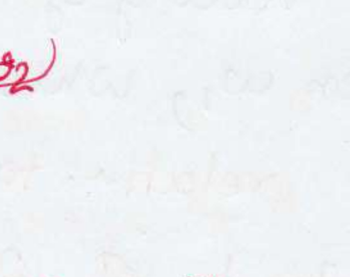
$\dot{j} \times \hat{k}$

v_{j}

$\cancel{\frac{di}{dt}}$



$\underline{\omega_2 \times (\omega_1)}\cdot k$



$\omega_T = \omega_i + \omega_j$

$$\frac{d\omega_T}{dt} = \frac{d\omega_1}{dt}i + \omega\frac{di}{dt}$$

$$+\frac{d\omega_2}{dt}j + \omega_2\frac{dj}{dt}$$

$\omega_1 \times$ $\quad$ ~~$\omega_1 \hat{i} \times \omega_2$~~

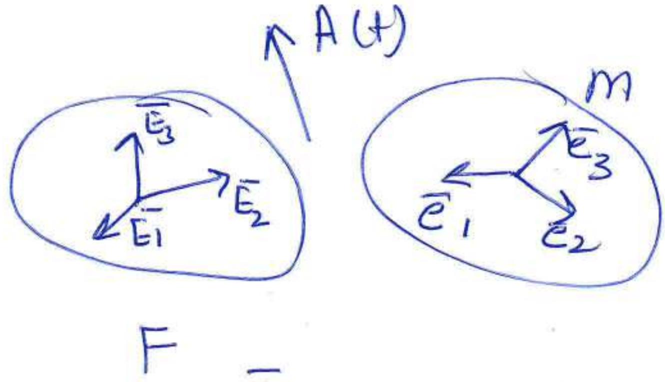
$$\omega_1[\underline{\omega_T \times j}] + \omega_2(\omega_T\,\hat{i})$$

$$\omega_1 j \times (\omega_1 \underline{i} + \omega j) \times j \qquad \omega_2 \times \left(\frac{dj}{dt}\right)$$

$$\omega_1\,\underline{\omega}\,\hat{k}$$

(A) (B)

Angular Velocity of a Frame m relative F



consider a right handed unit triad $\bar{E}_i$ fixed to F and triad $\bar{e}_i$ fixed to m.

The rate of change of orientation of m relative to F at time t is governed by $\dot{\bar{e}}_i|_F$

Let

$$\left.\begin{aligned} \dot{\bar{e}}_{1F}(t) &= a_1 e_1(t) + a_2 e_2(t) + a_3 e_3(t) \\ \dot{\bar{e}}_{2F}(t) &= b_1 e_1(t) + b_2 e_2(t) + b_3 e_3(t) \\ \dot{\bar{e}}_{3F}(t) &= c_1 e_1(t) + c_2 e_2(t) + c_3 e_3(t) \end{aligned}\right\} \quad (9)$$

Nine components a_i, b_i, c_i are not independent since $\bar{e}_i$ are mutually orthogonal unit vectors.

Different w.r.t

$$e_1(t) \cdot e_1(t) = 1 \Rightarrow 2 e_1(t) \cdot \dot{e}_1(t) = 0$$

Now multiply first eq. of (9).

$$\dot{\bar{e}}_{1F} \cdot e_1(t) = a_1 + 0 + 0 \Rightarrow a_1 = 0$$

Similarly $e_2(t) \cdot e_2(t) = 1$, $e_3(t) \cdot e_3(t) = 1$, $b_2 = 0$, $c_3 = 0$

(B)

Now $\bar{e}_1(t) \cdot \bar{e}_2(t) = 0$

Differentiating w.r.t. time

$$\bar{e}_1(t) \cdot \dot{e}_2(t) + \dot{e}_1(t) e_2(t) = 0$$

$$b_1 + a_2 = 0 \Rightarrow b_1 = -a_2$$

Similarly

$$\bar{e}_1(t) \cdot \bar{e}_3(t) = 0 \,, \qquad \bar{e}_2(t) \cdot \bar{e}_3(t) = 0$$

Which gives.

$$c_2 = -b_3 \,, \quad a_3 = -c_1$$

The similar relations have been obtained by cyclic changes $a \to b \to c \to a$ and $1 \to 2 \to 3 \to 1$

Thus only three components a_2, b_3, c_1 are independent. Rewrite eq. (1).

$$\dot{\bar{e}}_{1F}(t) = (a_2 \bar{e}_2 - c_1 \bar{e}_3) = [b$$

$$= (b_3 \bar{e}_1 + c_1 \bar{e}_2 + a_2 \bar{e}_3) \times \bar{e}_1$$

$$\Rightarrow c_1 \bar{e}_2 \times \bar{e}_1 + a_2 \bar{e}_3 \times \bar{e}_1 \Rightarrow (-c_1 \bar{e}_3 + a_2 \bar{e}_2)$$

$$\boxed{\dot{\bar{e}}_{1F}(t) = \underline{\omega} \times \bar{e}_1}$$

$$\dot{\bar{e}}_{2F}(t) = -a_2 e_1 + b_3 e_3 = [b_3\bar{e}_1 + c_1\bar{e}_2 + a_2\bar{e}_3] \times \bar{e}_2$$

$$\Rightarrow \underline{\omega} \times \bar{e}_2$$

$$\mathring{\bar{e}}_{3F}(t) = c_1 e_1 - b_3 e_2 = [b_3\hat{e}_1 + c_1\bar{e}_2 + a_2\bar{e}_3] \times \bar{e}_3$$

$$= \omega \times \bar{e}_3$$

i.e $\quad \mathring{\bar{e}}_{iF}(t) = \underline{\omega} \times \bar{e}_i \qquad \text{———} \quad (P)$

Now solve for $\underline{\omega}$ by forming $\bar{e}_i \times (P)$

$$\bar{e}_i \times \mathring{\bar{e}}_{iF} = \underline{\bar{e}_i \times (\omega \times \bar{e}_i)} \Rightarrow (e_i \cdot e_i)\omega - (e_i \cdot \omega) e_i$$

$$= \cancel{[\bar{e}_1\bar{e}_1 + \bar{e}_2\bar{e}_2 + \bar{e}_3\bar{e}_3]\,\omega = (e_1 e_1 \cdot \omega + e_2 \cdot \omega e_2 + e_3 \cdot \omega e_3)}$$

$$\cancel{3\omega - \omega = 2\omega.}$$

Now

$$\bar{e}_i . e_i = \bar{e}_1 \cdot \bar{e}_1 + \bar{e}_2 \cdot \bar{e}_2 + \bar{e}_3 \cdot \bar{e}_3 = 3$$

$$(\underline{\bar{e}_i \cdot \omega}) \cdot e_i \Rightarrow \omega_i e_i = \omega_1 e_1 + \omega_2 e_2 + \omega_3 e_3 = \omega$$

$$\bar{e}_i \times \mathring{\bar{e}}_{iF}(t) = 3\omega - \omega = 2\omega$$

$$\boxed{\omega = \frac{1}{2}\bar{e}_i \times \mathring{\bar{e}}_{iF}}$$

Angular velocity $\omega_{m/F}$ of frame m relative to frame F

D

Relation between $\mathring{A}_{Im}$ and $\mathring{A}_{IF}$.

Let $A_i(t)$ and $a_i(t)$ be the components of vector $\underline{A}(t)$ relative to $\bar{E}_i$ & $\bar{e}_i$

$$\bar{A}(t) = A_1(t)\bar{E}_1 + A_2(t)\bar{E}_2 + A_3(t)\bar{E}_3 = a_1(t)\bar{e}_1 + a_2(t)\bar{e}_2 + a_3(t)\bar{e}_3$$

$$= A_i(t)\bar{E}_i = a_i(t)e_i$$

$$\mathring{\bar{A}}_{/F} = \dot{A}_i E_i = \text{constant} = \dot{A}_i E_i$$

$$\mathring{A}_{lm} = \mathring{A}_{le_i} = \text{constant} = \dot{a}_i e_i$$

Now $$\mathring{A}_{IF} = (\overset{\circ}{\overline{a_i e_i}})_{IF} = \dot{a}_i \bar{e}_i + a_i \dot{\bar{e}}_i |_F$$

$$\mathring{A}_{IF} = \mathring{A}_{Im} + a_i \bar{\omega} \times \bar{e}_i = \mathring{A}_{Im} + \omega \times a_i e_i$$

$$\mathring{A}_{IF} = \mathring{A}_{Im} + \omega_{m/F} \times \underline{A}$$

$$\boxed{(\ )_{IF} = (\ )_{Im} + \omega_{m/F} \times (\ )}$$

In particular, if $\bar{A}$ is constant vector in m

then $\bar{A}_{Im} = 0$ $$\boxed{\mathring{A}_F = \omega \times A}$$

Note

ω = independent of the location or the direction of e_i in m. It is a measure of the rate of change of orientation of the frame m as a whole relative to the frame F and not a particular line element in it.

E

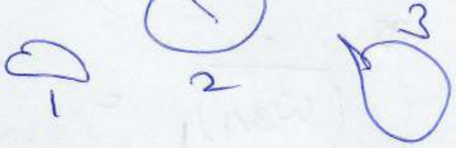
Using the equation Q.,

time derivative of $\omega_{m/F}$ relative to frame

$$\boxed{\overset{\circ}{\bar{\omega}}_F = \dot{\bar{\omega}}_m + \bar{\omega} \times \omega = \overset{\circ}{\bar{\omega}}_m}$$

$\dot{\omega} = \dot{\bar{\omega}}_{/F} = \dot{\omega}_{/m}$ = angular acceleration of frame m relative to frame F.

<u>Example</u>: Let $\bar{\omega}_{3|1}$, $\bar{\omega}_{3|2}$, $\bar{\omega}_{2|1}$ be the angular velocities of frame 3 relative to frame 1, frame 3 relative to fram 2 and frame 2 relative to frame 1



$$\cancel{\dot{A}_{|1} = \dot{A}_{|3} + \omega_{3|1} \times \bar{A} =}$$

$$\dot{A}_{|1} = \dot{A}_{|3} + \omega_{3|1} \times \bar{A} = \dot{A}_{|2} + \omega_{2|1} \times \bar{A} = \cancel{A}$$

$$= \left(\dot{A}_{|3} + \omega_{3/2} \times \bar{A}\right) + \omega_{2|1} \times \bar{A}$$

$$= \bar{A}_{|3} + (\omega_{3/2} + \omega_{2/1}) \times \bar{A}$$

$$\omega_{3/1} = \omega_{3/2} + \omega_{2/1}$$

Differentiating

$$\dot{\bar{\omega}}_{3|1} = \dot{\bar{\omega}}_{3/2} + \dot{\bar{\omega}}_{2|1} \Rightarrow \overset{\circ}{\bar{\omega}}_{3/2} + \omega_{2|1} \times \omega_{3/2} + \dot{\omega}_{2/1}$$

F

Fram (1), Fram (2) = Frame (3)

$\omega_{3/2} = 10$ rad/sec

$\omega_{2/1} = 5$ rad/see

$\underline{\omega_{3/1}} = ?\ \omega_{3/2} + \omega_{2/1}$

$\omega_{3/1} = \omega_{3/2} + \omega_{2/1} =$

$\cancel{\dot{\overline{\omega}}_{3/1} = \dot{\overline{\omega}}_{3/2} + \omega_{2/1} \times \omega_{3/2} + (\omega_{2/1})}$

$\dot{\overline{(\omega_{3/1})}}_1 = \dot{\overline{(\omega_{3/2})}}|_1 + \overline{(\omega_{2/1})}|_1$

$\dot{\overline{(\omega_{3/1})}}_1 = \dot{\overline{\omega}}_{3/2}|_1 + \underline{\omega_{2/1} \times \omega_{3/2}} + \dot{\overline{\omega}}_{2/1}$

Now double derivative

$$\boxed{\begin{aligned}\left(\ddot{\overline{\omega}}_{3/1}\right)\Big|_1 &= \left(\dot{\overline{\omega}}_{3/2}\right)_1 + \left(\dot{\overline{\omega}}_{2/1}\right)_1 \times \omega_{3/2} \\ &\quad + \omega_{2/2} \times \left(\dot{\overline{\omega}}_{3/2}\right)_1 + \ddot{\overline{\omega}}_{2/1}\Big|_1 \\ \left(\dot{\omega}_{3/1}\right)_1 &= \ddot{\overline{\omega}}_{3/2}\Big|_1\end{aligned}}$$

Example 15.1

A disc C is mounted on a shaft AB in Fig. 15.9. The shaft and disc rotate with a constant angular speed ω_2 of 10 rad/sec relative to the platform to which bearings A and B are attached. Meanwhile, the platform rotates at a constant angular speed ω_1 of 5 rad/sec relative to the ground in a direction parallel to the Z axis of the ground reference XYZ. What is the angular velocity vector $\boldsymbol{\omega}$ for the disc C relative to XYZ? What are $(d\boldsymbol{\omega}/dt)_{XYZ}$ and $(d^2\boldsymbol{\omega}/dt^2)_{XYZ}$?

The total angular velocity $\boldsymbol{\omega}$ of the disc relative to the ground is easily given at all times as follows:

$$\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 \text{ rad/sec} \tag{a}$$

At the instant of interest as depicted by Fig. 15.9, we have for $\boldsymbol{\omega}$:

$$\boldsymbol{\omega} = 5\boldsymbol{k} + 10\boldsymbol{j} \text{ rad/sec}$$

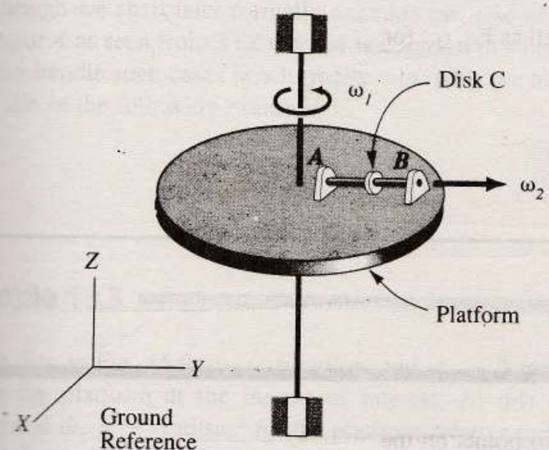


Figure 15.9. Rotating disc on rotating platform.

To get the first time derivative of $\boldsymbol{\omega}$, we go back to Eq. (a), which is always valid and hence can be differentiated with respect to time. Using a dot to represent the time derivative as seen from *XYZ*, we have

$$\dot{\boldsymbol{\omega}} = \dot{\boldsymbol{\omega}}_1 + \dot{\boldsymbol{\omega}}_2 \tag{b}$$

Consider now the vector $\boldsymbol{\omega}_2$. Note that this vector is constrained in direction to be always collinear with the axis *AB* of the bearings of the shaft. This clearly is a physical requirement. Also, since ω_2 is of constant value, we may think of the vector $\boldsymbol{\omega}_2$ as *fixed* to the platform along *AB*. Therefore, since the platform has an angular velocity of $\boldsymbol{\omega}_1$ relative to *XYZ*, we can say:

$$\dot{\boldsymbol{\omega}}_2 = \boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2 \tag{c}$$

Example 15.1 (Continued)

As for $\dot{\boldsymbol{\omega}}_1$, namely the other vector in Eq. (b), we note that as seen from XYZ, $\boldsymbol{\omega}_1$ is a constant vector and so at all times $\dot{\boldsymbol{\omega}}_1 = \mathbf{0}$. Hence Eq. (b) can be written as follows:

$$\dot{\boldsymbol{\omega}} = \boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2 \qquad \text{from (c)} \qquad \text{(d)}$$

This equation is valid at all times and so can be differentiated again. At the instant of interest as depicted by Fig. 15.9, we have for $\dot{\boldsymbol{\omega}}$:

$$\dot{\boldsymbol{\omega}} = 5\boldsymbol{k} \times 10\boldsymbol{j} = \mathbf{-50}\boldsymbol{i}\ \textbf{rad/sec}^2 \qquad \text{(e)}$$

To get $\ddot{\boldsymbol{\omega}}$, we now differentiate (d) with respect to time. We have

$$\begin{aligned}\ddot{\boldsymbol{\omega}} &= \dot{\boldsymbol{\omega}}_1 \times \boldsymbol{\omega}_2 + \boldsymbol{\omega}_1 \times \dot{\boldsymbol{\omega}}_2 \\ &= \mathbf{0} + \boldsymbol{\omega}_1 \times (\boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2)\end{aligned} \qquad \text{(f)}$$

where we have used the fact that $\dot{\boldsymbol{\omega}}_1 = \mathbf{0}$ at all times as well as Eq. (c) for $\dot{\boldsymbol{\omega}}_2$. At the instant of interest, we have

$$\ddot{\boldsymbol{\omega}} = 5\boldsymbol{k} \times (5\boldsymbol{k} \times 10\boldsymbol{j}) = \mathbf{-250}\boldsymbol{j}\ \textbf{rad/sec}^3$$

Example 15.2

In Example 15.1, consider a position vector $\boldsymbol{\rho}$ between two points on the rotating disc (see Fig. 15.10). The length of $\boldsymbol{\rho}$ is 100 mm and, at the instant of interest, is in the vertical direction. What are the first and second time derivatives of $\boldsymbol{\rho}$ at this instant as seen from the ground reference?

It should be obvious that the vector $\boldsymbol{\rho}$ is fixed to the disc which has at all times an angular velocity relative to XYZ equal to $\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$. Hence, at all times we can say:

$$\dot{\boldsymbol{\rho}} = \overbrace{(\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2)}^{\text{w.r.t ground}} \times \boldsymbol{\rho} \qquad \text{(a)}$$

Figure 15.10. Displacement vector in disc.

At the instant of interest, we have noting that $\boldsymbol{\rho} = 100\boldsymbol{k}$

$$\dot{\boldsymbol{\rho}} = (5\boldsymbol{k} + 10\boldsymbol{j}) \times 100\boldsymbol{k} = \mathbf{1{,}000}\boldsymbol{i}\ \textbf{mm/sec} \qquad \text{(b)}$$

To get the second derivative of $\boldsymbol{\rho}$, go back to Eq. (a) and differentiate:

$$\ddot{\boldsymbol{\rho}} = (\dot{\boldsymbol{\omega}}_1 + \dot{\boldsymbol{\omega}}_2) \times \boldsymbol{\rho} + (\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2) \times \dot{\boldsymbol{\rho}}$$

Example 15.2 (Continued)

Noting that $\dot{\boldsymbol{\omega}}_1 = \mathbf{0}$ at all times and, as discussed in Example 15.1, that $\boldsymbol{\omega}_2$ is fixed in the platform, we can say:

$$\ddot{\boldsymbol{\rho}} = (\mathbf{0} + \boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2) \times \boldsymbol{\rho} + (\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2) \times \dot{\boldsymbol{\rho}} \qquad \text{(c)}$$

At the instant of interest we have, on noting Eq. (b):

$$\ddot{\boldsymbol{\rho}} = (5\boldsymbol{k} \times 10\boldsymbol{j}) \times 100\boldsymbol{k} + (5\boldsymbol{k} + 10\boldsymbol{j}) \times 1{,}000\boldsymbol{i} \text{ mm/sec}^2$$

$$\ddot{\boldsymbol{\rho}} = 10\boldsymbol{j} - 10\boldsymbol{k} \text{ m/sec}^2$$

Although we shall later formally examine the case of the time derivative of vector $\boldsymbol{A}$ as seen from XYZ when $\boldsymbol{A}$ is *not fixed* in a body or a reference frame, we can handle such cases less formally with what we already know. We illustrate this in the following example.

Example 15.3

For the disc in Fig. 15.9, $\omega_2 = 6$ rad/sec and $\dot{\omega}_2 = 2$ rad/sec^2, both relative to the platform at the instant of interest. At this instant, $\omega_1 = 2$ rad/sec and $\dot{\omega}_1 = -3$ rad/sec^2 for the platform relative to the ground. Find the angular acceleration vector $\dot{\boldsymbol{\omega}}$ for the disc relative to the ground at the instant of interest.

The angular velocity of the disc relative to the ground at all times is

$$\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 \qquad \text{(a)}$$

For $\dot{\boldsymbol{\omega}}$, we can then say

$$\dot{\boldsymbol{\omega}} = \dot{\boldsymbol{\omega}}_1 + \dot{\boldsymbol{\omega}}_2 \qquad \text{(b)}$$

It is apparent on inspecting Fig. 15.11 that at all times $\boldsymbol{\omega}_1$ is vertical, and so we can say:

$$\dot{\boldsymbol{\omega}}_1 = \frac{d}{dt}_{XYZ}(\omega_1 \boldsymbol{k}) = \dot{\omega}_1 \boldsymbol{k} \qquad \text{(c)}$$

Example 15.3 (Continued)

However, $\boldsymbol{\omega}_2$ is changing direction and, most importantly, is changing magnitude. Because of the latter, $\boldsymbol{\omega}_2$ cannot be considered fixed in a reference or a rigid body for purposes of computing $\dot{\boldsymbol{\omega}}_2$. To get around this difficulty, we fix a unit vector $\boldsymbol{j}'$ *onto the platform* to be collinear with the centerline of the shaft *AB* as shown in Fig. 15.11. We know the angular velocity of this unit vector; it is $\boldsymbol{\omega}_1$ at all times. We can then express $\boldsymbol{\omega}_2$ in the following manner, which is valid at all times:

$$\boldsymbol{\omega}_2 = \omega_2 \boldsymbol{j}' \qquad \text{(d)}$$

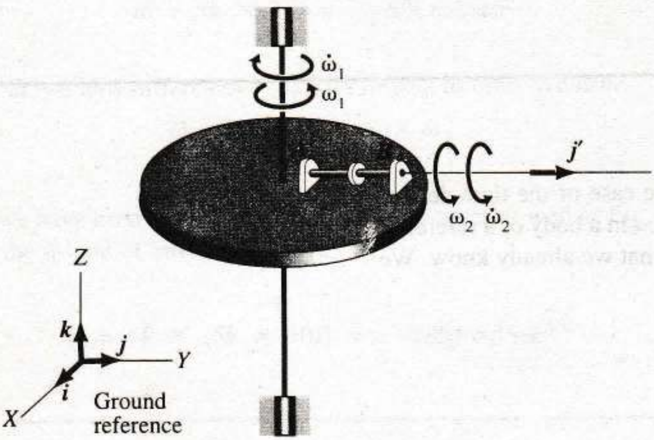


Figure 15.11. Unit vector $\boldsymbol{j}'$ fixed to platform.

We can differentiate the above with respect to time as follows:

$$\dot{\boldsymbol{\omega}}_2 = \dot{\omega}_2 \boldsymbol{j}' + \omega_2 \dot{\boldsymbol{j}}'$$

But $\boldsymbol{j}'$ is *fixed* to the platform which has angular velocity $\boldsymbol{\omega}_1$ relative to *XYZ* at all times. Hence, we have for the above,

$$\dot{\boldsymbol{\omega}}_2 = \dot{\omega}_2 \boldsymbol{j}' + \omega_2(\boldsymbol{\omega}_1 \times \boldsymbol{j}') \qquad \text{(e)}$$

w.r.t.

Thus, Eq. (b) then can be given as

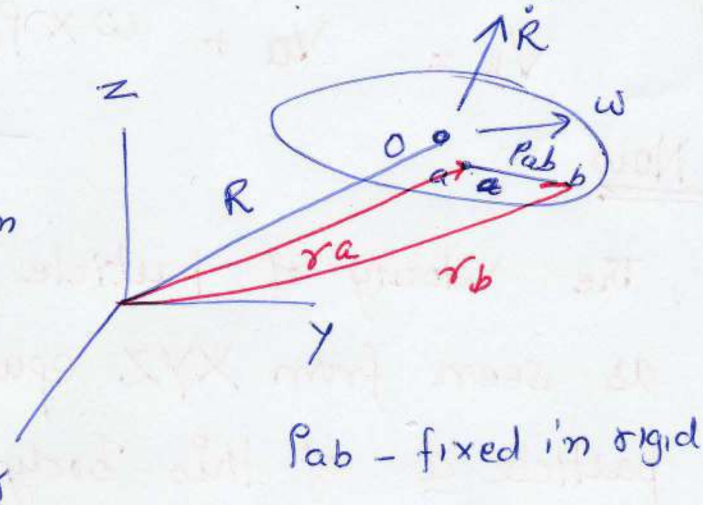
$$\dot{\boldsymbol{\omega}} = \dot{\omega}_1 \boldsymbol{k} + \dot{\omega}_2 \boldsymbol{j}' + \omega_2(\boldsymbol{\omega}_1 \times \boldsymbol{j}')$$

This expression is valid at all times and could be differentiated again. At the instant of interest, we can say, noting that $\boldsymbol{j}' = \boldsymbol{j}$ at this instant,

$$\dot{\boldsymbol{\omega}} = -3\boldsymbol{k} + 2\boldsymbol{j} + 6(2\boldsymbol{k} \times \boldsymbol{j})$$

$$\dot{\boldsymbol{\omega}} = -12\boldsymbol{i} + 2\boldsymbol{j} - 3\boldsymbol{k} \text{ rad/sec}^2$$

Application of the fixed-vector concept:



From Reference XYZ, a vector A fixed in a rigid body or fixed in xyz, the time derivatives can be obtained.

$$\overset{\circ}{A} = \omega \times A$$

ρ_{ab} - fixed in rigid body.

Now applying the above result,

vector ρ_{ab} is fixed in rigid body.

The body has translational velocity $\overset{\circ}{R}$ relative to XYZ corresponding to some point O and angular velocity ω relative to XYZ with the axis of rotation going through O. Now we can write

$$\overset{\circ}{\rho}_{ab} = \omega \times \rho_{ab}$$

Now consider the position vectors at a & b as shown,

$$r_a + \rho_{ab} = r_b$$

Take time derivative as seen from XYZ

$$\left(\frac{dr_a}{dt}\right)_{XYZ} + \left(\frac{d\rho_{ab}}{dt}\right)_{XYZ} = \left(\frac{dr_b}{dt}\right)_{XYZ}$$

$$\left(\frac{d\rho_{ab}}{dt}\right) = V_b - V_a \quad \Bigg| \quad V_a = \left(\frac{dr_a}{dt}\right)_{XYZ}$$

2

$$V_b = V_a + \omega \times \rho_{ab}$$ (use correct sequene $\rho_{ab} = -\rho_{ba}$)

——— (1)

Now

The velocity of particle b of a rigid body as seen from XYZ equals the velocity of particle a of this body as seen from XYZ plus the veloity of particle b relative to particle a

Differentiating eq (1) w.r.t. time will give the relationship of accelerations of two points

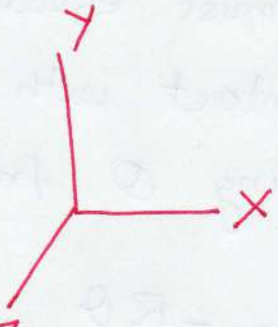
$$a_b = a_a + \dot{\omega} \times \rho_{ab} + \omega \times \dot{\rho}_{ab}$$

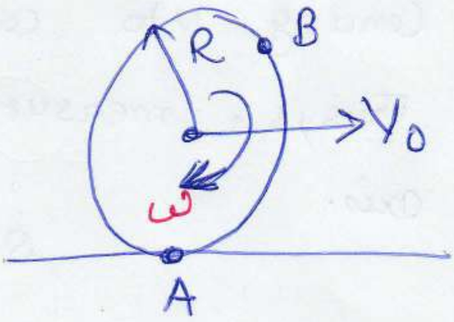
$$a_b = a_a + \dot{\omega} \times \rho_{ab} + \omega \times (\omega \times \rho_{ab})$$

Motion of two points of a rigid body as seen from a single reference.

Special case

Circular cylinder rolling without ~~split~~ slipping.





Point of contact:

A of the cylinder with the ground has instantaneously zero velocity and hence we have pure instantaneous rotation at any time tabout an instantaneous axis of rotation at the line of contact.

Velocity of Point B.

$$V_B = V_A + \omega \times \rho_{AB}$$

$$V_B = \omega \times \rho_{AB}$$

From this equation, it is clear that computing the velocity of any point on the cylinder, we can think of the cylinder as hinged at the point of contact.

In particular $V_0 = \omega \hat{k} \times R\hat{j} \Rightarrow -\omega R\hat{i}$ $(\because \rho_{AO} =$

if the velocity V_0 is known, angular velocity has a magnitude of V_0/R

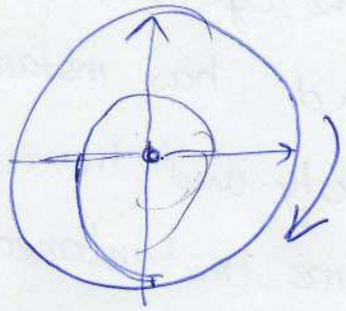
$i \times j$
$j \times k$
$k \times i$

Another way

V & ω is to realize that the distance S that O moves must equal the length of circumference coming into contact with the ground.

Thus is, measuring θ from the x axis to the Y axis.

$$S = -R\theta$$



$$V_O = -R\dot{\theta} = -R\omega$$

$$a_O = -R\ddot{\theta} = -R\alpha$$

Now

$$a_O = a_a + \dot{\omega} \times \rho_{AO} + \omega \times (\omega \times \rho_{AO})$$

$$-R\ddot{\theta}\hat{i} = a_A + \ddot{\theta}\underline{k} \times R\hat{\underline{j}} + \dot{\theta}k \times (\dot{\theta}k \times R\hat{j})$$

$$\cancel{-R\ddot{\theta}\hat{i}} = a_A \cancel{- R\ddot{\theta}\hat{i}} - R\dot{\theta}^2\hat{j}$$

$$\boxed{a_A = R\dot{\theta}^2\hat{j}}$$

V. Imp Conclusion.

Point A is accelerating upward toward the center of the cylinder.

Next, let us determine the acceleration vector for the *point of contact A* of the cylinder. Thus, we can say for points A and O:

$$\boldsymbol{a}_O = \boldsymbol{a}_A + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho}_{AO} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}_{AO})$$

Therefore,

$$-R\ddot{\theta}\boldsymbol{i} = \boldsymbol{a}_A + \ddot{\theta}\boldsymbol{k} \times R\boldsymbol{j} + \dot{\theta}\boldsymbol{k} \times (\dot{\theta}\boldsymbol{k} \times R\boldsymbol{j}) \tag{15.9}$$

Carrying out the products:

$$-R\ddot{\theta}\boldsymbol{i} = \boldsymbol{a}_A - R\ddot{\theta}\boldsymbol{i} - R\dot{\theta}^2\boldsymbol{j}$$

Therefore, cancelling terms, we get

$$\boldsymbol{a}_A = R\dot{\theta}^2\boldsymbol{j} \tag{15.10}$$

We see that *point A is accelerating upward toward the center of the cylinder.*[5] This information will be valuable for us in Chapter 16 when we study rigid-body dynamics.

[5]This conclusion must apply also to a sphere rolling without slipping on a flat surface. As for acceleration of other points of the cylinder, we do not have a simple formula but must insert data for these points into the acceleration formula valid for two points of a rigid body.

Example 15.4

Wheel D rotates at an angular speed ω_1 of 2 rad/sec counterclockwise in Fig. 15.15. Find the angular speed ω_E of gear E relative to the ground at the instant shown in the diagram.

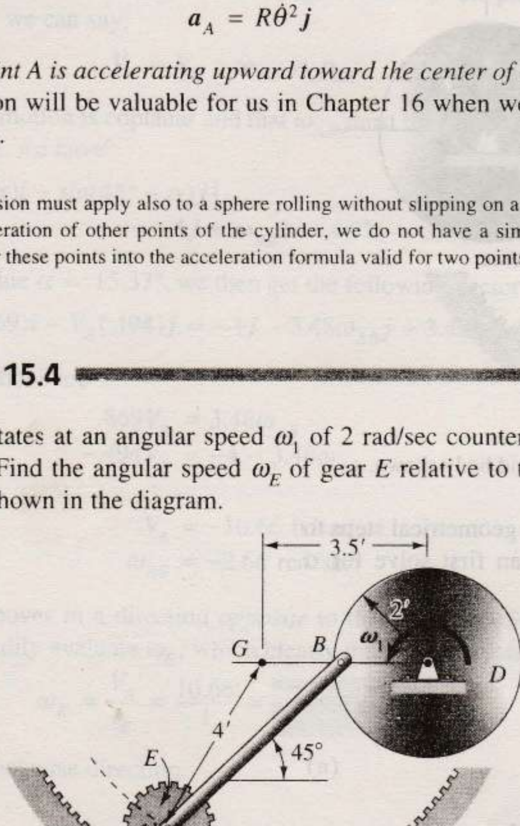


Figure 15.15. Two-dimensional device.

We have information about two points of one of the rigid bodies, namely AB, of the device. At B, the velocity must be downward with the value of $(\omega_1)(r_D) = 4$ ft/sec as shown in Fig. 15.16. Furthermore, since point A must travel a circular path of radius GA we know that A has velocity V_A with a direction at right angles to GA. Accordingly, since the angle between GA and the vertical is $(90° - 45° - \alpha) = (45° - \alpha)$ as can readily be seen on inspecting Fig. 15.16, then the angle between V_A and the horizontal must also be $(45° - \alpha)$ because of the *mutual perpendicularity* of the sides of these angles. If we can determine velocity V_A, we can get the desired angular speed of gear A immediately.

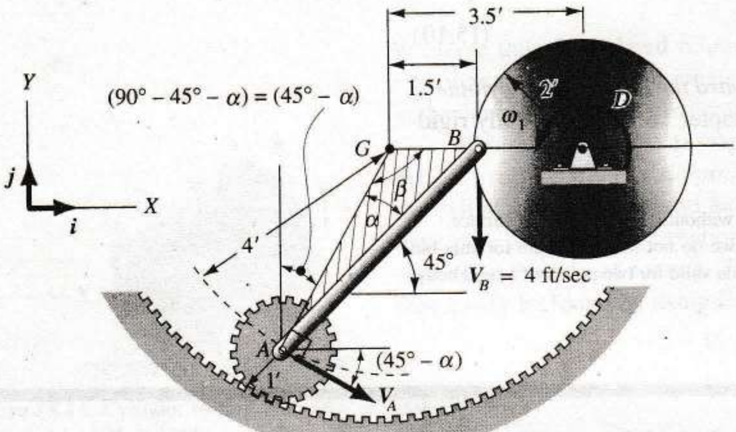


Figure 15.16. Velocity vectors for two points of a rigid body shown.

Before examining rigid body AB, we have some geometrical steps to take. Considering triangle GAB in Fig. 15.16, we can first solve for α using the law of sines as follows:

$$\frac{GA}{\sin(\measuredangle GBA)} = \frac{GB}{\sin \alpha}$$

Therefore, since $\measuredangle GBA = 45°$

$$\frac{4}{\sin 45°} = \frac{1.5}{\sin \alpha} \tag{a}$$

Solving for α, we get

$$\alpha = 15.37° \tag{b}$$

The angle β is then easily evaluated considering the angles in the triangle GBA. Thus.

$$\begin{aligned}\beta &= 180° - \alpha - \measuredangle GBA \\ &= 180° - 15.37° - 45° = 119.6°\end{aligned} \tag{c}$$

Example 15.4 (Continued)

Finally, we can determine AB of the triangle, again using the law of sines. Thus,

$$\frac{AB}{\sin\beta} = \frac{GA}{\sin 45^\circ}$$
$$\frac{AB}{\sin 119.6^\circ} = \frac{4}{.707}$$

Solving for AB, we get

$$AB = 4.92 \text{ ft} \tag{d}$$

We now can consider bar AB as our rigid body. For the points A and B on this body, we can say:

$$\boldsymbol{V}_A = \boldsymbol{V}_B + \boldsymbol{\omega}_{AB} \times \boldsymbol{\rho}_{BA}$$

Noting that the motion is coplanar and that $\boldsymbol{\omega}_{AB}$ must then be normal to the plane of motion, we have[6]

$$V_A[\cos(45^\circ - \alpha)\boldsymbol{i} - \sin(45^\circ - \alpha)\boldsymbol{j}]$$
$$= -4\boldsymbol{j} + \omega_{AB}\boldsymbol{k} \times 4.92(-\cos 45^\circ \boldsymbol{i} - \sin 45^\circ \boldsymbol{j})$$

Inserting the value $\alpha = 15.37^\circ$, we then get the following vector equation:

$$V_A(.869)\boldsymbol{i} - V_A(.494)\boldsymbol{j} = -4\boldsymbol{j} - 3.48\omega_{AB}\boldsymbol{j} + 3.48\omega_{AB}\boldsymbol{i} \tag{e}$$

The scalar equations are

$$\begin{aligned} .869V_A &= 3.48\omega_{AB} \\ -.494V_A &= -4 - 3.48\omega_{AB} \end{aligned} \tag{f}$$

Solving, we get[7]

$$\begin{aligned} V_A &= -10.66 \text{ ft/sec} \\ \omega_{AB} &= -2.66 \text{ rad/sec} \end{aligned} \tag{g}$$

Thus, point A moves in a direction *opposite* to that shown in Fig. 15.16. We now can readily evaluate ω_E, which clearly must have a value of

$$\omega_E = \frac{V_A}{r_E} = \frac{10.66}{1} = \boxed{10.66 \text{ rad/sec}}$$

in the counterclockwise direction.

[6]Our practice will be to consider unknown angular velocities as *positive*. The sign for the unknown angular velocity coming out of the computations will then correspond to the actual *convention* sign for the angular velocity.

[7]By having assumed ω_{AB} as positive and thus *counterclockwise* for the reference xy employed, we conclude from the presence of the minus sign that the assumption is wrong and that ω_{AB} must be *clockwise* for the reference used. It is significant to note that as a result of the initial positive assumption, the result $\omega_{AB} = -2.66$ rad/sec gives at the same time the *correct convention sign* for the actual angular velocity for the reference used.

Example 15.5

In the device in Fig. 15.17, find the angular velocities and angular accelerations of both bars.

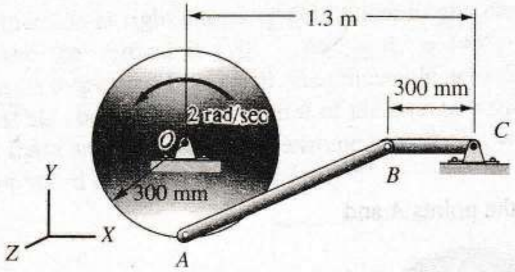


Figure 15.17. Two-dimensional device.

We shall consider points A and B of bar AB. Note first that at the instant shown:

$$\begin{aligned} \boldsymbol{V}_B &= -(.300)(\omega_{BC})\boldsymbol{j} \text{ m/sec} && \text{(a)} \\ \boldsymbol{V}_A &= (2)(.300)\boldsymbol{i} \\ &= .600\boldsymbol{i} \text{ m/sec} && \text{(b)} \end{aligned}$$

Noting that $\boldsymbol{\omega}_{AB}$ must be oriented in the Z direction because we have plane motion in the XY plane, we have for Eq. 15.6:

$$\begin{aligned} \boldsymbol{V}_B &= \boldsymbol{V}_A + \boldsymbol{\omega}_{AB} \times \boldsymbol{\rho}_{AB} \\ -.300\omega_{BC}\boldsymbol{j} &= .600\boldsymbol{i} + (\omega_{AB}\boldsymbol{k}) \times (\boldsymbol{i} + .300\boldsymbol{j}) && \text{(c)} \\ -.300\omega_{BC}\boldsymbol{j} &= .600\boldsymbol{i} + \omega_{AB}\boldsymbol{j} - .300\omega_{AB}\boldsymbol{i} \end{aligned}$$

Note we have assumed ω_{BC} and ω_{AB} as positive and thus counterclockwise. The scalar equations are:

$$\begin{aligned} .600 &= .300\omega_{AB} \\ -.300\omega_{BC} &= \omega_{AB} && \text{(d)} \end{aligned}$$

We then get

$$\begin{aligned} \omega_{AB} &= 2 \text{ rad/sec} \\ \omega_{BC} &= -6.67 \text{ rad/sec} && \text{(e)} \end{aligned}$$

Therefore, ω_{AB} is counterclockwise while ω_{BC} must be clockwise.

Let us now turn to the angular acceleration considerations for the bars. We consider separately now points A and B of bar AB. Thus,

$$\begin{aligned} \boldsymbol{a}_A &= (r\omega^2)\boldsymbol{j} = (.300)(2^2)\boldsymbol{j} = 1.200\boldsymbol{j} \text{ m/sec}^2 \\ \boldsymbol{a}_B &= \rho_{BC}\omega_{BC}^2\boldsymbol{i} + \rho_{BC}\dot{\omega}_{BC}(-\boldsymbol{j}) \\ &= (.300)(-6.67^2)\boldsymbol{i} - .300\dot{\omega}_{BC}\boldsymbol{j} \\ &= 13.33\boldsymbol{i} - .300\dot{\omega}_{BC}\boldsymbol{j} \end{aligned}$$

Example 15.5 (Continued)

Again, we have assumed $\dot{\omega}_{BC}$ positive and thus counterclockwise. Considering bar AB, we can say for Eq. 15.7:

$$\boldsymbol{a}_B = \boldsymbol{a}_A + \dot{\boldsymbol{\omega}}_{AB} \times \boldsymbol{\rho}_{AB} + \boldsymbol{\omega}_{AB} \times (\boldsymbol{\omega}_{AB} \times \boldsymbol{\rho}_{AB}) \quad \text{(f)}$$

Noting that $\dot{\boldsymbol{\omega}}_{AB}$ must be in the Z direction, we have for the foregoing equation:

$$\begin{aligned} &13.33\boldsymbol{i} - .300\dot{\omega}_{BC}\boldsymbol{j} \\ &\quad = 1.200\boldsymbol{j} + \dot{\omega}_{AB}\boldsymbol{k} \times (\boldsymbol{i} + .300\boldsymbol{j}) + (2\boldsymbol{k}) \times \left[2\boldsymbol{k} \times (\boldsymbol{i} + .300\boldsymbol{j})\right] \quad \text{(g)} \end{aligned}$$

The scalar equations are

$$\begin{aligned} 17.33 &= -.300\dot{\omega}_{AB} \\ -.300\dot{\omega}_{BC} &= \dot{\omega}_{AB} \end{aligned}$$

We get

$$\begin{aligned} \dot{\omega}_{AB} &= -57.8 \text{ rad/sec}^2 \\ \dot{\omega}_{BC} &= 192.6 \text{ rad/sec}^2 \end{aligned}$$

Clearly, for the reference used, $\dot{\omega}_{AB}$ must be clockwise and $\dot{\omega}_{BC}$ must be counterclockwise.

Example 15.6

(a) In Example 15.5, find the *instantaneous axis of rotation* for the rod *AB*.

The intersection of the instantaneous axis of rotation with the *xy* plane will be a point *E* in a hypothetical rigid-body extension of bar *AB* having zero velocity at the instant of interest. We can accordingly say:

$$\boldsymbol{V}_E = \boldsymbol{V}_A + \boldsymbol{\omega}_{AB} \times \boldsymbol{\rho}_{AE}$$

Therefore,

$$\boldsymbol{0} = .60\boldsymbol{i} + (2\boldsymbol{k}) \times (\Delta x\boldsymbol{i} + \Delta y\boldsymbol{j}) \quad \text{(a)}$$

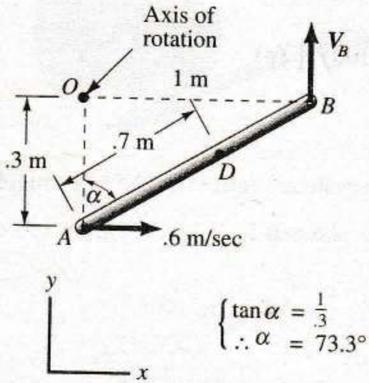
where Δx and Δy are the components of the directed line segment from point *A* to the center of rotation *E*. The scalar equations are:

$$\begin{aligned} 0 &= .60 - 2\Delta y \\ 0 &= 2\Delta x \end{aligned}$$

Clearly, $\Delta y = .3$ and $\Delta x = 0$. Thus, the center of rotation is point *O*.

Example 15.6 (Continued)

We could have easily deduced this result by inspection in this case. The velocity of each point of bar AB must be at *right angles* to a line from the center of rotation to the point. The velocity of point A is in the horizontal direction and the velocity of point B is in the vertical direction. Clearly, as seen from Fig. 15.18, point O is the only point from which lines to points A and B are normal to the velocities at these points.



$$\begin{cases} \tan\alpha = \frac{1}{.3} \\ \therefore \alpha = 73.3° \end{cases}$$

Figure 15.18. Instantaneous axis of rotation of AB.

(b) Now using the instantaneous axis of rotation, find the magnitudes of the velocity and acceleration of point D (Fig. 15.18) using data from the previous example.

In Fig. 15.19, we show the velocity vector normal to line OD. Using the law of cosines for triangle AOD, we can find OD which is a key distance for this example. Thus noting from Fig. 15.18 that $\alpha = 73.3°$, we have

$$\overline{OD} = \left[.7^2 + .3^2 - (2)(.7)(.3)(\cos 73.3°)\right]^{1/2} = .6777 \text{ m}$$

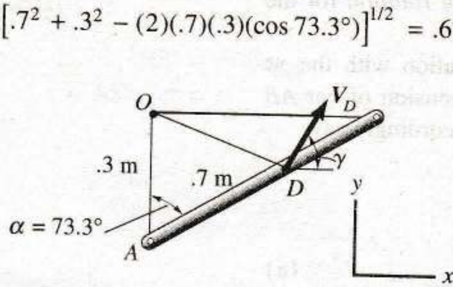


Figure 15.19. Velocity vector for point D.

We then say from rotational motion about the instantaneous center of rotation O,

$$V_D = (.6777)(\omega_{AB}) = (.6777)(2) = 1.355 \text{ m/s}$$

Example 15.6 (Continued)

For the acceleration, we have (see Fig. 15.20)

$$a_D = \left[(a_D)_c^2 + (a_D)_t^2\right]^{1/2}$$

where $(a_D)_c$ and $(a_D)_t$, respectively, are the centripetal and tangential components of acceleration at point D. Noting that r for point D is .6777 m, we get for the above

$$\therefore a_D = \left\{\left(\frac{V_D^2}{r}\right)^2 + \left[(r)(\dot{\omega}_{AB})\right]^2\right\}^{1/2}$$

$$= \left\{\left(\frac{1.355^2}{.6777}\right)^2 + \left[(.6777)(57.8)\right]^2\right\}^{1/2} = \boxed{39.26 \text{ m/s}^2} \qquad \text{(b)}$$

We now get the vectors $\mathbf{V}_D$ and $\boldsymbol{a}_D$. For this purpose we determine the angle β of the tinted triangle in Fig. 15.20 by first using the law of sines for triangle AOD

$$\frac{.7}{\sin(90° - \beta)} = \frac{.6777}{\sin 73.3°}$$

$$\therefore \beta = 8.373°$$

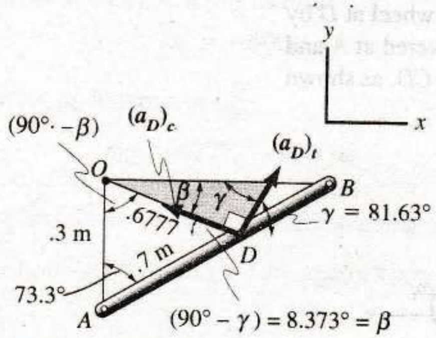


Figure 15.20. Acceleration components of point D.

Hence, looking at the tinted triangle it is clear that $\gamma = 90° - 8.373° = 81.63°$. We can now give $\mathbf{V}_D$ (see Fig. 15.19).

$$\mathbf{V}_D = V_D(\cos\gamma\boldsymbol{i} + \sin\gamma\boldsymbol{j}) = 1.355(\cos 81.63\boldsymbol{i} + \sin 81.63\boldsymbol{j})$$

$$\boxed{\mathbf{V}_D = .1972\boldsymbol{i} + 1.341\boldsymbol{j} \text{ m/s}}$$

Example 15.6 (Continued)

For the acceleration vector, we refer back to Eq. (b) for components of $\boldsymbol{a}_D$. Noting Fig. 15.20, we have

$$\boldsymbol{a}_D = r(\dot{\omega}_{AB})[\cos\gamma \boldsymbol{i} + \sin\gamma \boldsymbol{j}] + \frac{V_D^2}{r}[-\cos\beta \boldsymbol{i} + \sin\beta \boldsymbol{j})$$
$$= (.6777)(-57.8)[\cos 81.63°\boldsymbol{i} + \sin 81.63°\boldsymbol{j}] + \frac{1.355^2}{.6777}[-\cos 8.373°\boldsymbol{i} + \sin 8.373°\boldsymbol{j}]$$

$$\therefore \quad \boxed{\boldsymbol{a}_D = -8.38\boldsymbol{i} - 38.36\boldsymbol{j} \text{ m/s}^2}$$

*Example 15.7

A disk E is rotating about a fixed axis HG at a constant angular speed ω_1 of 5 rad/sec in Fig. 15.21. A bar CD is held by the wheel at D by a ball-joint connection and is guided along a rod AB cantilevered at A and B by a collar at C having a second ball-joint connection with CD, as shown in the diagram. Compute the velocity of C.

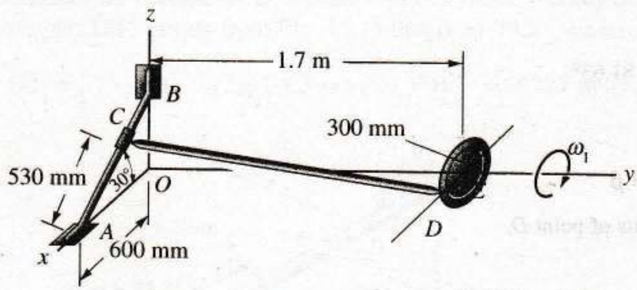


Figure 15.21. Three-dimensional device.

We shall need the vector $\boldsymbol{\rho}_{DC}$. Thus,

$$\boldsymbol{\rho}_{DC} = \boldsymbol{r}_C - \boldsymbol{r}_D$$
$$= [(.600 - .530\cos 30°)\boldsymbol{i} + .530\sin 30°\boldsymbol{k}] - (1.7\boldsymbol{j} + .300\boldsymbol{i})$$
$$= -.1590\boldsymbol{i} - 1.7\boldsymbol{j} + .265\boldsymbol{k} \text{ m}$$

Example 15.7 (Continued)

Now employ Eq. 15.6 for rod CD. Thus,

$$\boldsymbol{V}_C = \boldsymbol{V}_D + \boldsymbol{\omega}_{CD} \times \boldsymbol{\rho}_{DC}$$

Therefore, assuming C is going from B to A

$$V_C(\cos 30°\boldsymbol{i} - \sin 30°\boldsymbol{k}) = (5)(.30)\boldsymbol{k} + (\omega_x\boldsymbol{i} + \omega_y\boldsymbol{j} + \omega_z\boldsymbol{k}) \times (-.1590\boldsymbol{i} - 1.7\boldsymbol{j} + .265\boldsymbol{k})$$

$$V_C(.866\boldsymbol{i} - .500\boldsymbol{k}) = 1.50\boldsymbol{k} - 1.7\omega_x\boldsymbol{k} - .265\omega_x\boldsymbol{j} + .1590\omega_y\boldsymbol{k} + .265\omega_y\boldsymbol{i} - .1590\omega_z\boldsymbol{j} + 1.7\omega_z\boldsymbol{i})$$

The scalar equations are:

$$.866V_C = .265\omega_y + 1.7\omega_z \quad \text{(a)}$$
$$0 = -.265\omega_x - .1590\omega_z \quad \text{(b)}$$
$$-.500V_C = 1.50 - 1.7\omega_x + .1590\omega_y \quad \text{(c)}$$

From these equations, we cannot solve for ω_x, ω_y, and ω_z because the spin of CD about its own axis (allowed by the ball joints) can have *any value* without affecting the velocity of slider C. However, we can determine V_C, as we shall now demonstrate.

In Eq. (b), solve for ω_x in terms of ω_z.

$$\omega_x = -.600\omega_z \quad \text{(d)}$$

In Eq. (a), solve for ω_y in terms of ω_z:

$$\omega_y = 3.27V_C - 6.415\omega_z \quad \text{(e)}$$

Substitute for ω_x and ω_y in Eq. (c) using the foregoing results:

$$-.500V_C = 1.50 - (1.7)(-.600\omega_z) + (.1590)(3.27V_C - 6.415\omega_z)$$

Therefore,

$$-1.020V_C = 1.5 + 1.020\omega_z - 1.020\omega_z$$
$$V_C = -1.471 \text{ m/sec}$$

Hence,

$$\boldsymbol{V}_c = -1.471(\cos 30°\boldsymbol{i} - \sin 30°\boldsymbol{k})$$

$$\boldsymbol{V}_c = -1.274\boldsymbol{i} + .7355\boldsymbol{k} \text{ m/sec}$$

Clearly, contrary to our assumption C is going from A to B.

15.6 General Relationship Between Time Derivatives of a Vector for Different References

In Section 15.4, we considered the time derivatives of a vector A "fixed" in a reference xyz moving arbitrarily relative to XYZ. Our conclusions were:

$$\left(\frac{dA}{dt}\right)_{xyz} = \mathbf{0}$$

$$\left(\frac{dA}{dt}\right)_{XYZ} = \boldsymbol{\omega} \times \boldsymbol{A}$$

We now wish to extend these considerations to include time derivatives of a vector $\boldsymbol{A}$ which is not necessarily fixed in reference xyz. Primarily, our intention in this section is to relate time derivatives of such vectors $\boldsymbol{A}$ as seen both from reference xyz and from XYZ, two references moving arbitrarily relative to each other.

For this purpose, consider Fig. 15.24, where we show a moving particle P with a position vector $\boldsymbol{\rho}$ in reference xyz. Reference xyz moves arbitrarily relative to reference XYZ with translational velocity $\dot{\boldsymbol{R}}$ and angular velocity $\boldsymbol{\omega}$ in accordance with **Chasles' theorem.** We shall now form a relation between $(d\boldsymbol{\rho}/dt)_{xyz}$ and $(d\boldsymbol{\rho}/dt)_{XYZ}$. We shall then extend this result so as to relate the time derivative of any vector $\boldsymbol{A}$ as seen from any two references.

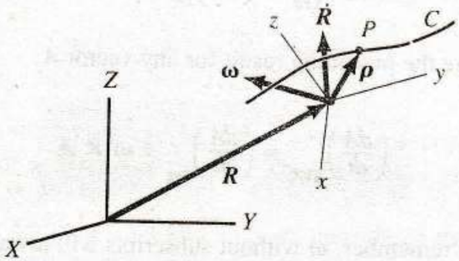


Figure 15.24. xyz moves relative to XYZ.

To reach the desired results effectively, we shall express the vector $\boldsymbol{\rho}$ in terms of components parallel to the xyz reference:

$$\boldsymbol{\rho} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} \tag{15.11}$$

where $\boldsymbol{i}, \boldsymbol{j}$, and $\boldsymbol{k}$ are unit vectors for reference xyz. Differentiating this equation with respect to time for the xyz reference, we have:[8]

$$\left(\frac{d\boldsymbol{\rho}}{dt}\right)_{xyz} = \dot{x}\boldsymbol{i} + \dot{y}\boldsymbol{j} + \dot{z}\boldsymbol{k} \tag{15.12}$$

[8]Note that $\dot{x}$, $\dot{y}$, and $\dot{z}$ are time derivatives of scalars and accordingly there is no identification with any reference as far as the time derivative operation is concerned.

If we next take the derivative of $\boldsymbol{\rho}$ with respect to time for the XYZ reference, we must remember that $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ of Eq. 15.11 generally will each be a function of time, since these vectors will generally have some rotational motion relative to the XYZ reference. Thus, if dots are used for the time derivatives:

$$\left(\frac{d\boldsymbol{\rho}}{dt}\right)_{XYZ} = (\dot{x}\boldsymbol{i} + \dot{y}\boldsymbol{j} + \dot{z}\boldsymbol{k}) + (x\dot{\boldsymbol{i}} + y\dot{\boldsymbol{j}} + z\dot{\boldsymbol{k}}) \tag{15.13}$$

The unit vector $\boldsymbol{i}$ is a vector *fixed* in reference xyz, and accordingly $\dot{\boldsymbol{i}}$ equals $\boldsymbol{\omega} \times \boldsymbol{i}$. The same conclusions apply to $\boldsymbol{j}$ and $\boldsymbol{k}$. The last expression in parentheses can then be stated as

$$\begin{aligned} (x\dot{\boldsymbol{i}} + y\dot{\boldsymbol{j}} + z\dot{\boldsymbol{k}}) &= x(\boldsymbol{\omega} \times \boldsymbol{i}) + y(\boldsymbol{\omega} \times \boldsymbol{j}) + z(\boldsymbol{\omega} \times \boldsymbol{k}) \\ &= \boldsymbol{\omega} \times (x\boldsymbol{i}) + \boldsymbol{\omega} \times (y\boldsymbol{j}) + \boldsymbol{\omega} \times (z\boldsymbol{k}) \\ &= \boldsymbol{\omega} \times (x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}) = \boldsymbol{\omega} \times \boldsymbol{\rho} \end{aligned} \tag{15.14}$$

In Eq. 15.13 we can replace $(\dot{x}\boldsymbol{i} + \dot{y}\boldsymbol{j} + \dot{z}\boldsymbol{k})$ by $(d\boldsymbol{\rho}/dt)_{xyz}$, in accordance with Eq. 15.12, and $(x\dot{\boldsymbol{i}} + y\dot{\boldsymbol{j}} + z\dot{\boldsymbol{k}})$ by $\boldsymbol{\omega} \times \boldsymbol{\rho}$, in accordance with Eq. 15.14. Hence,

$$\left(\frac{d\boldsymbol{\rho}}{dt}\right)_{XYZ} = \left(\frac{d\boldsymbol{\rho}}{dt}\right)_{xyz} + \boldsymbol{\omega} \times \boldsymbol{\rho} \tag{15.15}$$

We can generalize the preceding result for any vector $\boldsymbol{A}$:

$$\left(\frac{d\boldsymbol{A}}{dt}\right)_{XYZ} = \left(\frac{d\boldsymbol{A}}{dt}\right)_{xyz} + \boldsymbol{\omega} \times \boldsymbol{A} \tag{15.16}$$

where, you must remember, $\boldsymbol{\omega}$ without subscripts will always be the *angular velocity of the xyz reference relative to the XYZ reference.* Note that Eq. 15.1 is a special case of Eq. 15.16 since for $\boldsymbol{A}$ fixed in xyz, $(d\boldsymbol{A}/dt)_{xyz} = \boldsymbol{0}$. We shall have much use for this relationship in succeeding sections.

15.7 Relationship Between Velocities of a Particle for Different References

We shall now define the velocity of a particle again in the presence of several references:

> ***The velocity of a particle relative to a reference is the derivative as seen from this reference of the position vector of the particle in the reference.***