Indian Institute of Technology Guwahati

ME 101: Engineering Mechanics (2016-2017, Sem II)

Quiz-2 Solution(3.04.2017)

Time: 8:00 AM – 8:45 AM Full Marks: 60

Q1. The acceleration of a particle in cylindrical coordinate system (r, θ, z) is given by

$$\boldsymbol{\vec{a}} = \left(\boldsymbol{\ddot{r}} - \boldsymbol{r}\,\boldsymbol{\dot{\theta}^2}\right)\boldsymbol{\hat{e}_r} + \left(\boldsymbol{r}\,\boldsymbol{\ddot{\theta}} + \ \boldsymbol{2}\,\boldsymbol{\dot{r}}\,\boldsymbol{\dot{\theta}}\right)\boldsymbol{\hat{e}_\theta} + \boldsymbol{\ddot{z}}\,\boldsymbol{\hat{e}_z}$$

where $\boldsymbol{\hat{e}_r}$, $\boldsymbol{\hat{e}_\theta}$ **and** $\boldsymbol{\hat{e}_z}$ are, respectively, the unit vectors in the instantaneous radial, azimuthal and axial directions of the particle. The rate of change of acceleration $\left(\boldsymbol{\frac{d\vec{a}}{dt}}\right)$ is called jerk.

(a) Derive an expression for the jerk of a particle in cylindrical polar coordinates.

(b) Calculate the jerk experienced by a small insect crawling with a constant speed of **1 cm/s** on a helical spring of **2 cm** diameter and 1 cm pitch. [15+5=20 marks]

Solution Q1-

(a)

$$\vec{a} = \left(\ddot{r} - r\,\dot{\theta}^2\right)\hat{e}_r + \left(r\,\ddot{\theta} + \ 2\,\dot{r}\,\dot{\theta}\right)\hat{e}_\theta + \ddot{z}\,\hat{e}_z$$

$$\frac{d\vec{a}}{dt} = \frac{d}{dt}\left[\left(\ddot{r} - r\,\dot{\theta}^2\right)\hat{e}_r\right] + \frac{d}{dt}\left[\left(r\,\ddot{\theta} + \ 2\,\dot{r}\,\dot{\theta}\right)\hat{e}_\theta\right] + \frac{d}{dt}\left[\ddot{z}\,\hat{e}_z\right]$$

$$= \frac{d}{dt}\left(\ddot{r} - r\,\dot{\theta}^2\right)\hat{e}_r + \left(\ddot{r} - r\,\dot{\theta}^2\right)\frac{d\hat{e}_r}{dt}$$

$$+ \frac{d}{dt}\left(r\,\ddot{\theta} + \ 2\,\dot{r}\,\dot{\theta}\right)\hat{e}_\theta + \left(r\,\ddot{\theta} + \ 2\,\dot{r}\,\dot{\theta}\right)\frac{d\hat{e}_\theta}{dt}$$

$$+ \frac{d\ddot{z}}{dt}\hat{e}_z + \ddot{z}\frac{d\hat{e}_z}{dt}$$

$$\frac{d}{dt}\left(\ddot{r} - r\,\dot{\theta}^2\right) = \dddot{r} - \dot{r}\dot{\theta}^2 - 2\,r\,\dot{\theta}\,\ddot{\theta}$$

$$\frac{d}{dt}\left(r\,\ddot{\theta} + \ 2\,\dot{r}\,\dot{\theta}\right) = \dot{r}\ddot{\theta} - r\,\dddot{\theta} + 2\,\ddot{r}\,\dot{\theta} + 2\,\dot{r}\,\ddot{\theta}$$

$$\left|\frac{d\hat{e}_r}{d\theta}\right| = \lim_{\Delta\theta\to 0}\left|\frac{\Delta\hat{e}_r}{\Delta\theta}\right| = \lim_{\Delta\theta\to 0}\left|\frac{2\sin\frac{\Delta\theta}{2}}{\Delta\theta}\right| = \lim_{\Delta\theta\to 0}\left|\frac{\sin\frac{\Delta\theta}{2}}{\frac{\Delta\theta}{2}}\right| = 1$$

$$\left|\frac{d\hat{e}_\theta}{d\theta}\right| = \lim_{\Delta\theta\to 0}\left|\frac{\Delta\hat{e}_\theta}{\Delta\theta}\right| = \lim_{\Delta\theta\to 0}\left|\frac{2\sin\frac{\Delta\theta}{2}}{\Delta\theta}\right| = \lim_{\Delta\theta\to 0}\left|\frac{\sin\frac{\Delta\theta}{2}}{\frac{\Delta\theta}{2}}\right| = 1$$

The directions of $\frac{d\hat{e}_r}{d\theta}$ and $\frac{d\hat{e}_\theta}{d\theta}$ are along increasing θ and decreasing r, respectively. Therefore,

$$\frac{d\hat{e}_r}{d\theta} = \hat{e}_\theta$$

$$\frac{d\hat{e}_\theta}{d\theta} = -\hat{e}_r$$

$$\frac{d\hat{e}_r}{dt} = \frac{d\hat{e}_r}{d\theta}\frac{d\theta}{dt} = \dot{\theta}\,\frac{d\hat{e}_r}{d\theta} = \dot{\theta}\,\hat{e}_\theta$$

$$\frac{d\hat{e}_\theta}{dt} = \frac{d\hat{e}_\theta}{d\theta}\frac{d\theta}{dt} = \dot{\theta}\,\frac{d\hat{e}_\theta}{d\theta} = -\dot{\theta}\,\hat{e}_r$$

$$\frac{d\vec{a}}{dt} = \frac{d}{dt}\left[\left(\ddot{r} - r\,\dot{\theta}^2\right)\hat{e}_r\right] + \frac{d}{dt}\left[\left(r\,\ddot{\theta} + \ 2\,\dot{r}\,\dot{\theta}\right)\hat{e}_\theta\right] + \frac{d}{dt}\left[\ddot{z}\,\hat{e}_z\right]$$

$$= \frac{d}{dt}\left(\ddot{r} - r\,\dot{\theta}^2\right)\hat{e}_r + \left(\ddot{r} - r\,\dot{\theta}^2\right)\frac{d\hat{e}_r}{dt}$$

$$+\frac{d}{dt}\left(r\,\ddot{\theta} + \ 2\,\dot{r}\,\dot{\theta}\right)\hat{e}_\theta + \left(r\,\ddot{\theta} + \ 2\,\dot{r}\,\dot{\theta}\right)\frac{d\hat{e}_\theta}{dt}$$

$$+\frac{d\ddot{z}}{dt}\hat{e}_z + \ddot{z}\frac{d\hat{e}_z}{dt}$$

$$\frac{d\vec{a}}{dt} = \left(\dddot{r} - \dot{r}\,\dot{\theta}^2 - 2\,r\,\dot{\theta}\,\ddot{\theta}\right)\hat{e}_r + \left(\ddot{r} - r\,\dot{\theta}^2\right)\dot{\theta}\,\hat{e}_\theta$$

$$+\left(\dot{r}\ddot{\theta} + r\,\dddot{\theta} + 2\,\ddot{r}\,\dot{\theta} \ + 2\,\dot{r}\,\ddot{\theta}\right)\hat{e}_\theta + \ \left(r\,\ddot{\theta} + \ 2\,\dot{r}\,\dot{\theta}\right)\left(-\dot{\theta}\,\hat{e}_r\right)$$

$$+\dddot{z}\,\hat{e}_z + 0$$

Collecting components along $\hat{e}_r$, $\hat{e}_\theta$ and $\hat{e}_z$ directions,

$$\frac{d\vec{a}}{dt} = \left(\dddot{r} - \dot{r}\,\dot{\theta}^2 - 2\,r\,\dot{\theta}\,\ddot{\theta} - r\,\dot{\theta}\,\ddot{\theta} - 2\,\dot{r}\,\dot{\theta}^2\right)\hat{e}_r$$

$$+\left(\ddot{r}\,\dot{\theta} \ - r\,\dot{\theta}^3 + \ \dot{r}\,\ddot{\theta} + \ r\,\dddot{\theta} + 2\,\ddot{r}\,\dot{\theta} \ + 2\,\dot{r}\,\ddot{\theta}\ \right)\hat{e}_\theta + \ \dddot{z}\,\hat{e}_z$$

$$\frac{d\vec{a}}{dt} = \left(\dddot{r} - 3\,\dot{r}\,\dot{\theta}^2 - 3\,r\,\dot{\theta}\,\ddot{\theta}\right)\hat{e}_r + \left(3\,\ddot{r}\,\dot{\theta} \ + \ 3\,\dot{r}\,\ddot{\theta} - r\,\dot{\theta}^3 + \ r\,\dddot{\theta}\ \right)\hat{e}_\theta + \ \dddot{z}\,\hat{e}_z$$

(b)

r = R = 0.01 m = const.,

$$\alpha = \tan^{-1}\left(\frac{P}{2\pi R}\right) = \tan^{-1}\left(\frac{0.01}{2\pi \times 0.01}\right) = 0.15915$$

$\dot{\theta} \ = \frac{v_\theta}{R} = \frac{v\cos\alpha}{R} = \ 0.987\ \text{s}^{-1} = \text{const.},$

$$\dot{z} \ = v_z = v\sin\alpha = \text{const.}$$

$$\frac{d\vec{a}}{dt} = \left(0 - 3\times 0\ \times\ \dot{\theta}^2 - 3\,R\times\dot{\theta}\ \times 0\right)\hat{e}_r + \left(3\times 0\times\dot{\theta}\ +\ 3\times 0\times 0 - R\,\dot{\theta}^3 + \ R\times 0\ \right)\hat{e}_\theta + \ 0\,\hat{e}_z$$

$$= 0\,\hat{e}_r + \left(-R\,\dot{\theta}^3\ \right)\hat{e}_\theta + \ 0\,\hat{e}_z$$

$$= -R\,\dot{\theta}^3\,\hat{e}_\theta$$

$$= -0.01\times\ (0.987)^3\,\hat{e}_\theta$$

$$= -0.00961\,\hat{e}_\theta\ \text{m/s}^3$$

Q2. At the instant shown in **Fig.1**, the rod **AB** is rotating about the **z** axis with an angular velocity $\boldsymbol{\omega_1} = \mathbf{4}\ \boldsymbol{rad}/s$ and an angular acceleration $\dot{\boldsymbol{\omega}}_\mathbf{1} = \mathbf{3}\ \boldsymbol{rad}/\boldsymbol{s}^\mathbf{2}$. At this same instant, the circular rod has an angular motion relative to the rod as shown. If the collar ***C*** is moving down around the circular rod with a speed of **7.6 cm**$/\boldsymbol{s}$, increasing at a rate $\mathbf{20.3}\ \boldsymbol{cm}/\boldsymbol{s}^\mathbf{2}$, both measured relative to the rod.
Determine (a) the collar's velocity and, (b) acceleration at this instant. [5+15=20 marks]

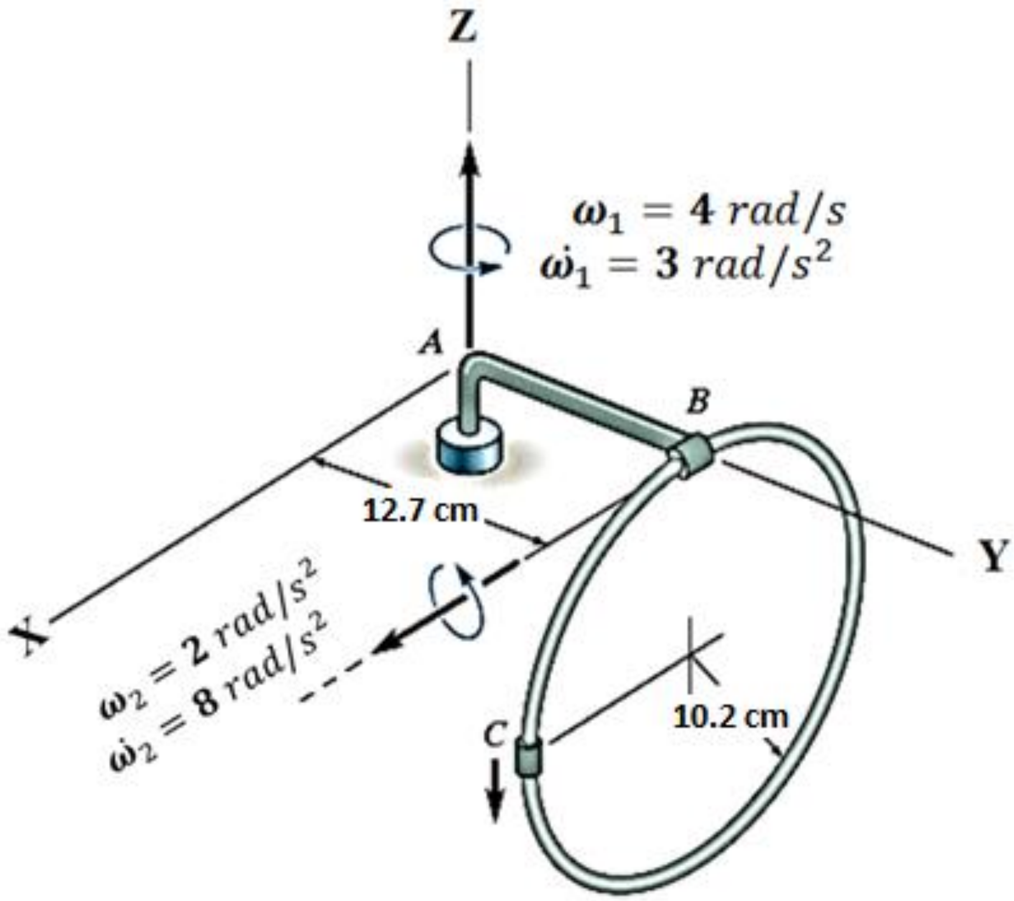


Fig. 1

SOLUTION:

$\boldsymbol{\Omega} = \{4\,\mathbf{k}\}\ \mathrm{rad/s}$

$\dot{\boldsymbol{\Omega}} = \{3\,\mathbf{k}\}\ \mathrm{rad/s^2}$

$\mathbf{r_B} = \{0.127\,\mathbf{j}\}\ \mathrm{m}$

$\mathbf{v_B} = (4\,\mathbf{k}) \times (0.127\,\mathbf{j}) = \{-0.508\,\mathbf{i}\}\ \mathrm{m/s}$

$$\mathbf{a_B} = \ddot{\mathbf{r}}_\mathbf{B} = \left[(\ddot{\mathbf{r}}_\mathbf{B})_{xyz} + \boldsymbol{\Omega} \times (\dot{\mathbf{r}}_\mathbf{B})_{xyz}\right] + \dot{\boldsymbol{\Omega}} \times \mathbf{r_B} + \boldsymbol{\Omega} \times \dot{\mathbf{r}}_\mathbf{B}$$

$$= (4\,\mathbf{k}) \times (-0.508\,\mathbf{i}) + (3\,\mathbf{k}) \times (0.127\,\mathbf{j})$$

$$= \{-0.381\,\mathbf{i} - 2.032\,\mathbf{j}\}\ \mathrm{m/s^2}$$

$\boldsymbol{\Omega}_{C/B} = \{2\,\mathbf{i}\}\ \mathrm{rad/s}$

$\dot{\boldsymbol{\Omega}}_{C/B} = \{8\,\mathbf{i}\}\ \mathrm{rad/s^2}$

$\mathbf{r}_{C/B} = \{0.1016\,\mathbf{i} - 0.1016\,\mathbf{k}\}\ \mathrm{m}$

$$(\mathbf{v}_{C/B})_{xyz} = (\dot{\mathbf{r}}_{C/B}) + \boldsymbol{\Omega}_{C/B} \times \mathbf{r}_{C/B}$$

$$= (-0.0762\,\mathbf{k}) + (2\,\mathbf{i}) \times (0.1016\,\mathbf{i} - 0.1016\,\mathbf{k})$$

$$= \{0.2032\,\mathbf{j} - 0.0762\,\mathbf{k}\}\ \mathrm{m/s}$$

$$(\mathbf{a}_{C/B})_{xyz} = \left[(\ddot{\mathbf{r}}_{C/B})_{xyz} + \boldsymbol{\Omega}_{C/B} \times (\dot{\mathbf{r}}_{C/B})_{xyz}\right] + \dot{\boldsymbol{\Omega}}_{C/B} \times \mathbf{r}_{C/B} + \boldsymbol{\Omega}_{C/B} \times \dot{\mathbf{r}}_{C/B}$$

$$= \left[\left(\frac{0.0762^2}{0.1016}\,\mathbf{i} - 0.2032\,\mathbf{k}\right) + (2\,\mathbf{i}) \times (-0.0762\,\mathbf{k})\right] + (8\,\mathbf{i}) \times (0.1016\,\mathbf{i} - 0.1016\,\mathbf{k}) + (2\,\mathbf{i}) \times (0.2032\,\mathbf{j} - 0.0762\,\mathbf{k})$$

$$= \{-0.05715\,\mathbf{i} + 1.1176\,\mathbf{j} + 0.2032\,\mathbf{k}\}\ \mathrm{m/s^2}$$

$$\mathbf{v_C} = \mathbf{v_B} + \boldsymbol{\Omega} \times \mathbf{r}_{C/B} + (\mathbf{v}_{C/B})_{xyz}$$

$$= (-0.508\,\mathbf{i}) + (4\,\mathbf{k}) \times (0.1016\,\mathbf{i} - 0.1016\,\mathbf{k}) + (0.2032\,\mathbf{j} - 0.0762\,\mathbf{k})$$

$$= \{-0.508\,\mathbf{i} + 0.6096\,\mathbf{j} - 0.0762\,\mathbf{k}\}\ \mathrm{m/s}$$

$$\mathbf{a} = \mathbf{a_B} + \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{C/B} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{C/B}) + 2\boldsymbol{\Omega} \times (\mathbf{v}_{C/B})_{xyz} + (\mathbf{a}_{C/B})_{xyz}$$

$$= (-0.381\,\mathbf{i} - 2.032\,\mathbf{j}) + (3\,\mathbf{k}) \times (0.1016\,\mathbf{i} - 0.1016\,\mathbf{k}) + (4\,\mathbf{k}) \times [(4\,\mathbf{k}) \times (0.1016\,\mathbf{i} - 0.1016\,\mathbf{k})] + 2(4\,\mathbf{k}) \times (0.2032\,\mathbf{j} - 0.0762\,\mathbf{k}) + (-0.05715\,\mathbf{i} + 1.1176\,\mathbf{j} + 0.2032\,\mathbf{k})$$

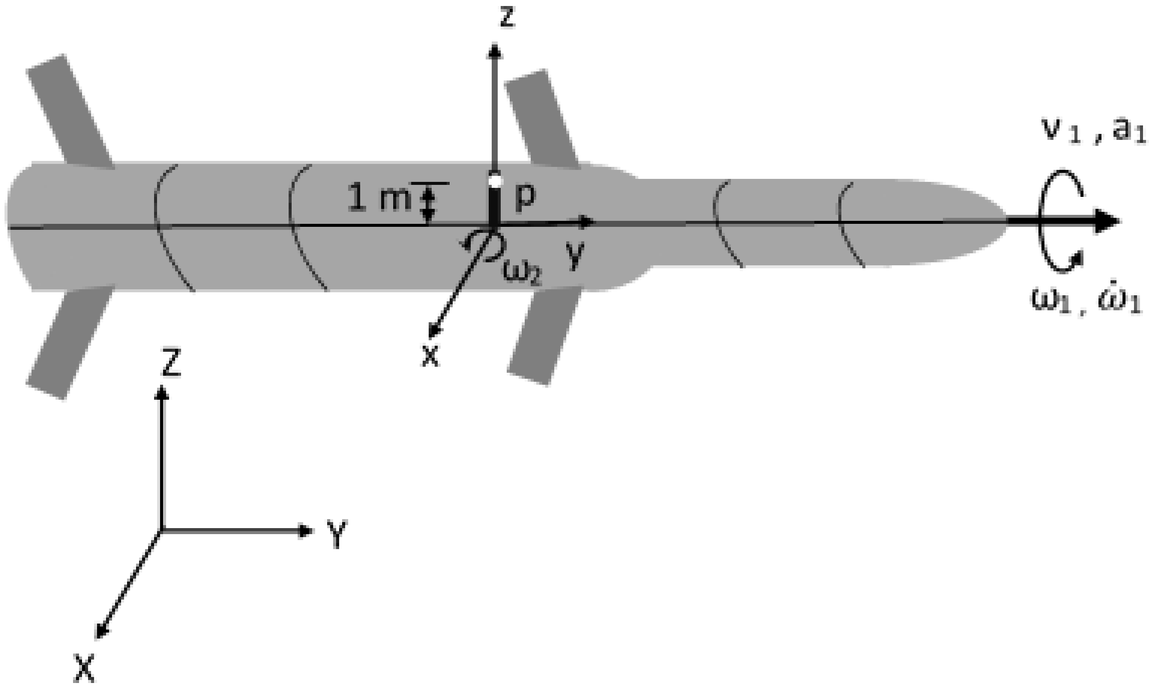
$$= \{-3.683\,\mathbf{i} - 0.6096\,\mathbf{j} + 0.2032\,\mathbf{k}\}\ \mathrm{m/s^2}$$

Q3. A missile travels in a straight line with respect to inertial reference XYZ at speed **v_1 = 4 m/s** and change of **speed a_1 = 2 m/s²** as shown in Fig. 2. At the same instant, the missile rolls about its direction of flight at an angular speed **ω_1 = 5 rad/s** and change in angular speed **$\dot{\omega}_1$= 5 rad/s²** as shown in the Fig. 2. Inside the missile **a rod is rotating at a constant angular speed (relative to the missile) ω_2 = 10 rad/s** about an axis perpendicular to the page**. A point p on the rod moves outward with speed 1 m/s with respect to the rod**. The point **p on the rod is at a distance of 1m from the axis of rotation**. Determine the following

(a) Find the angular velocity and angular acceleration of rod with respect to ground (XYZ).

(b) Find Coriolis acceleration $(2\vec{\omega}\times\vec{V})$ of Point p in the frame fixed with the rod and the frame fixed with the missile. [15+5=20 marks]

Fig 2.



Q3 - Fix frame 1 (XYZ) to the ground.

- Fix frame 2 (xyz) to the missile
- Fix frame 3 to the rode

~~Angular~~ Given

angular velocity of missile w.r.t ground $\underline{\omega}_2|_1 = \omega_1 \underline{j} = 5\underline{j}$ rad/s

angular acc^n of missile w.r.t ground $\underline{\dot{\omega}}_2|_1 = \dot{\omega}_1 \underline{j} = 5\underline{j}$ rad/s²

angular velocity of rod w.r.t missile $\underline{\omega}_3|_2 = \omega_2 \underline{i}$ rad/s $= 10\underline{i}$ rad/s

angular acceleration of rod w.r.t missile $\underline{\dot{\omega}}_3|_2 = \underline{0}$ rad/s²

velocity of missile w.r.t ground $\underline{v}_2|_1 = v_1 \underline{j} = 4\underline{j}$ m/s

acceleration of missile w.r.t ground $\underline{a}_2|_1 = a_1 \underline{j}$ $2\underline{j}$ m/s²

(a) Angular velocity of ~~of~~ rode w.r.t ground

$$\underline{\omega}_3|_1 = \underline{\omega}_3|_2 + \underline{\omega}_2|_1 \quad \text{—(1)}$$

$$\boxed{\omega_3|_1 = 10\underline{i} + 5\underline{j} \text{ rad/s}}$$

angular acceleration of rod w.r.t ground

$$\underline{\dot{\omega}}_3|_1 = \underline{\dot{\omega}}_{3/2}|_1 + \dot{\omega}_{2/1}|_1$$

$$= \dot{\omega}_3|_2 + \underline{\omega}_2|_1 \times \underline{\omega}_3|_2 + \underline{\dot{\omega}}_2|_1$$

$$= 0 + 5\underline{j} \times 10\underline{i} + 5\underline{j}$$

$$\boxed{\underline{\dot{\omega}}_3| = \underline{-50\,\underline{k} + 5\,\underline{j}}} \text{ rad/s}^2$$

(b) velocity of point p ~~is~~ with respect to rod

$$\underline{v}_{p/3} = 1\underline{k} \ m/s$$

angular velocity of rod w.r.t ground

$$\underline{\omega}_{3/1} = 10\underline{i} + 5\underline{j}$$

coriolis acceleration in frame attach to rod

$$\underline{a}_{cor} = 2\underline{\omega} \times v$$

$$= 2\underline{\omega}_{3/1} \times \underline{v}_{p/3}$$

$$= 2(10i + 5j) \times 1\underline{k}$$

$$\boxed{\underline{a}_{co} = -20\underline{j} + 10i} \ m/s^2$$

velocity of point p in frame attached to rod

$$\underline{v}_{p/2} = 1\underline{k} + \underline{\omega}_{3/2} \times \underline{r}$$

$$= 1\underline{k} + 10\underline{i} \times 1\underline{k}$$

$$= 1\underline{k} - 10\underline{j}$$

coriolis acceleration of p in frame attached to the missile

$$\underline{a}_{co} = 2\,\underline{\omega}_{2/1} \times \underline{v}_{p/2}$$

$$= 2 \times 5\underline{j} \times (1\underline{k} - 10\underline{j})$$

$$= 10\underline{i} - \underline{0}$$

$$\boxed{\underline{a}_{co} = 10\underline{i}} \ m/s^2$$