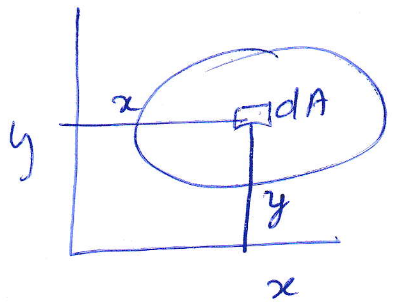
Moment of Inertia:

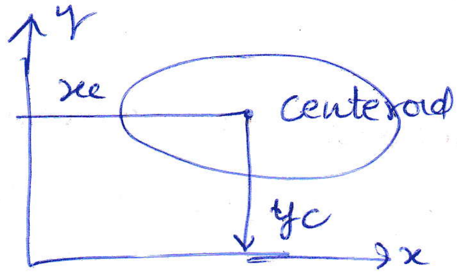
First moment of an Area:

$M_x = \int y\,dA$ about x

$M_y = \int_A x\,dA$ about y



Centroidal coordinates

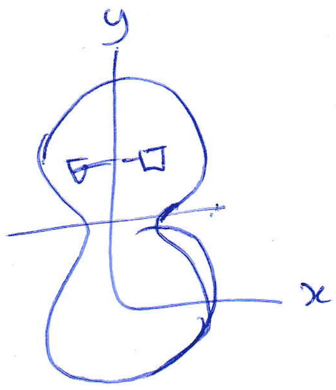


$$A y_c = \int_A y\,dA \Rightarrow y_c = \frac{\int_A y\,dA}{A} = \frac{M_x}{A}$$

$$x_c = \frac{\int_A x\,dA}{A}$$

axis of symmetry:

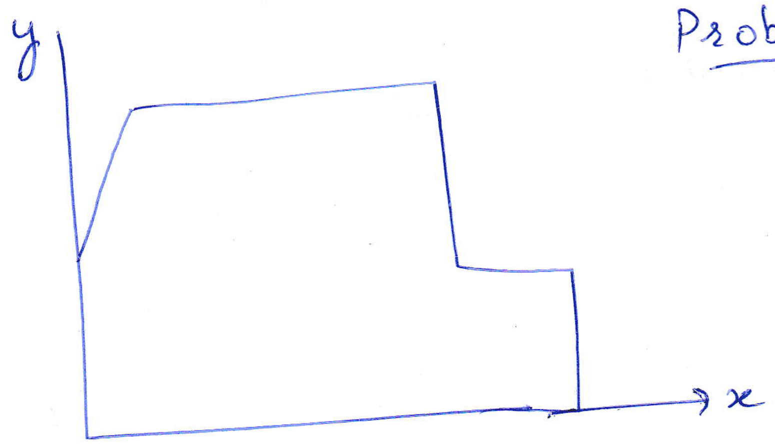
first moment of Area = 0



$x_c = 0$

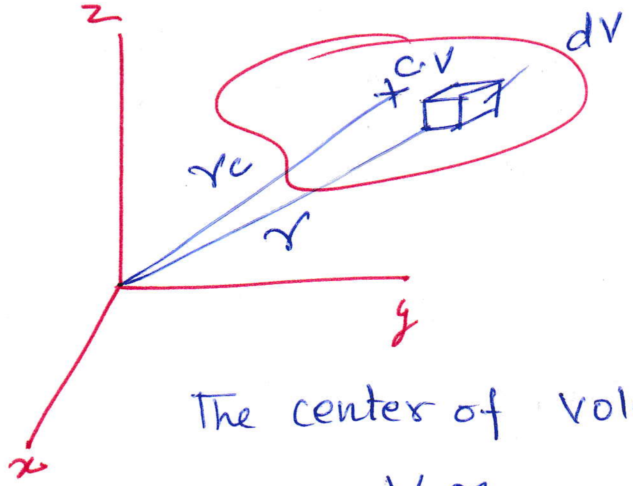
Orthogonal axes of symmetry: , For such area centroid can be easily determined.

Problem 8.2 :



Centroid of 3D bodies

First moment of volume, V, of a body about a point O where we have shown a reference xyz.



moment vector of volume

$$= \iiint_V r \, dV$$

The center of volume

$$V r_c = \iiint_V r \, dV$$

$$r_c = \frac{\iiint_V r \, dV}{V}$$

$$x_c = \frac{\iiint x \, dV}{\iiint dV}, \quad y_c = \frac{\iiint y \, dV}{\iiint dV}$$

$$z_c = \frac{\iiint z \, dV}{\iiint dV}$$

if We replace dv by $dm = \rho dV$

ρ = mass density

moment vector of mass = $\iiint r \rho dV$

$$M r_c = \iiint r \rho dV$$

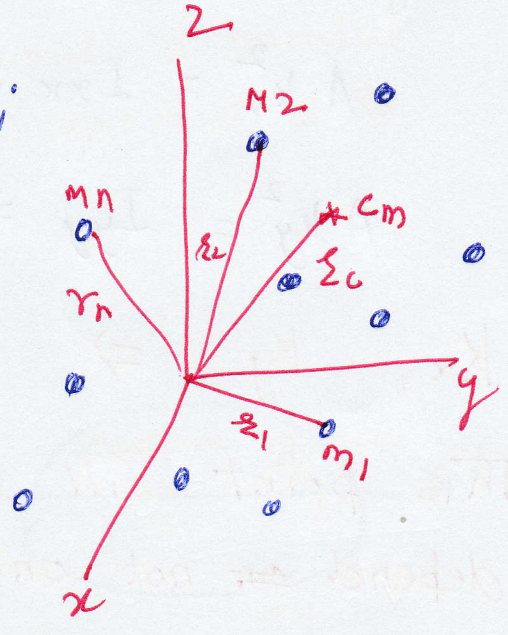
$$r_c = \frac{\iiint r \rho dV}{M}$$

continous body

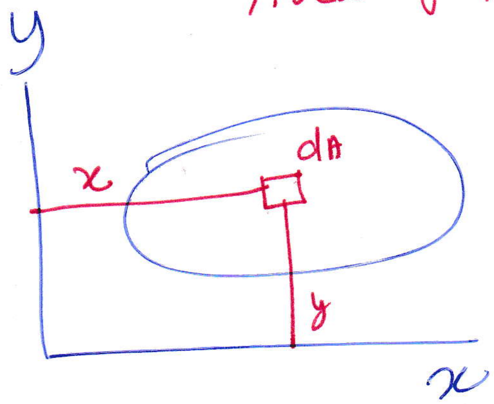
~~Syst~~ Descreti system: Center of mass of a system of n particles

$$\sum_{i=1}^{n} (m_i) r_c = \sum_{i=1}^{n} m_i r_i$$

$$r_c = \frac{\sum_{i=1}^{n} m_i r_i}{\sum_{i=1}^{n} m_i}$$



Second Moment & Product of Area of plane Area.



$$I_{xx} = \int_A y^2 \, dA$$

$$I_{yy} = \int_A x^2 \, dA$$

Note: 2nd moment of area can not be negative, incontrast to the first moment.

In an analogy to the centroid, the entire area may be concentrated at single point k_x, k_y

$$A k_x^2 = I_{xx} = \int_A y^2 \, dA \Rightarrow k_x^2 = \frac{\int_A y^2 \, dA}{A}$$

$$A k_y^2 = I_{yy} = \int_A x^2 \, dA \Rightarrow k_y^2 = \frac{\int_A x^2 \, dA}{A}$$

K_x, K_y $\Rightarrow$ radii of gyration.

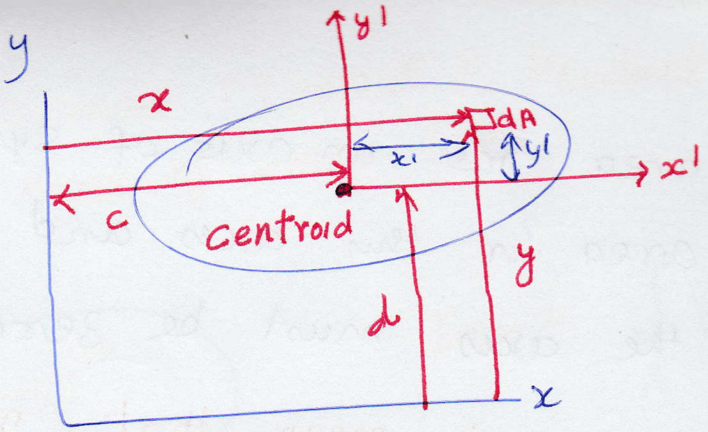
This point will have a position that depend ~~on~~ not only on the shape of the area but also on the position of the reference.

Wherea Centroid $\rightarrow$ Location is independent of the reference position

Product of Area

$$I_{xy} = \int_A xy \, dA$$

$\longrightarrow$ This may be negative



$$I_{xy} = \int_A xy\, dA = \int (x'+c)(y'+d)\, dA$$

$$I_{xy} = \underbrace{\int x'y'\,dA}_{= I'_{xy}} + \underbrace{\int cy'\,dA}_{= c\int y'\,dA = 0} + \underbrace{\int x'd\,dA}_{= c\int x'\,dA = 0} + \int cd\,dA$$

$$\boxed{I_{xy} = I'_{xy} + cdA}$$

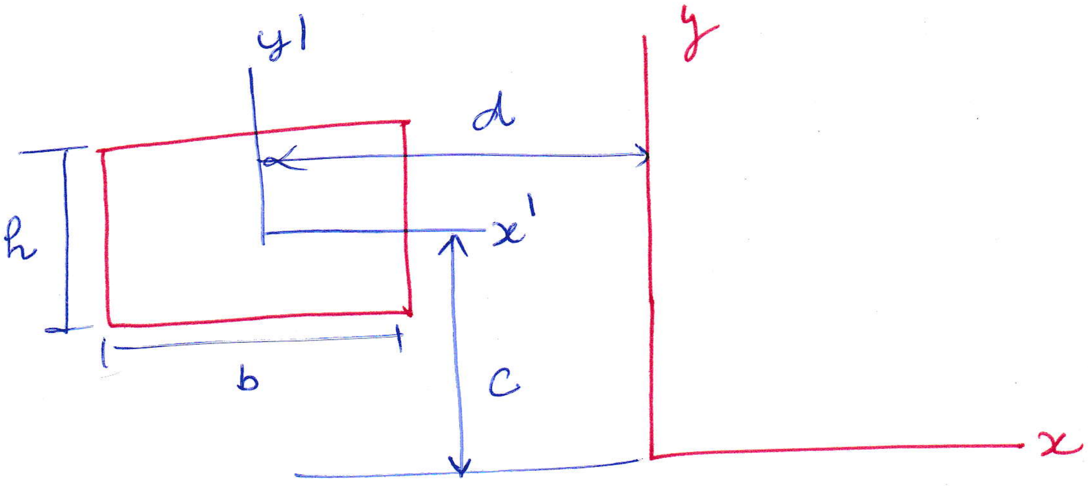
__Note__ c & $d \rightarrow$ measured from the xy axes to the centroid and must have appropriate sign

__Problem__ 8.6, 8.7, 8.8, 8.9

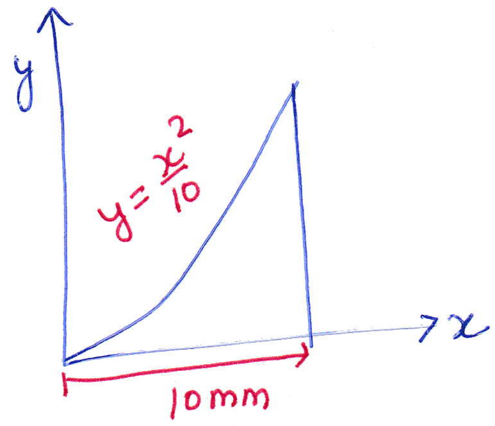
Ex 8.6

~~A secto~~

A rectangle is shown in Fig. Compute the second moments and product of area about the centroidal $x'y'$ axes as well as about the xy axes



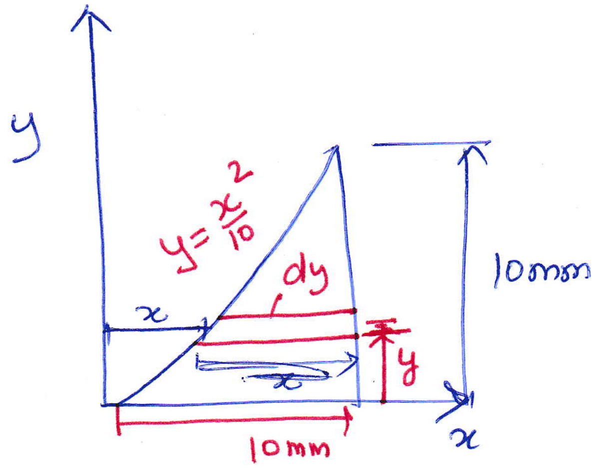
8.7 WHAT ARE I_{xx}, I_{yy}, I_{xy} for the area under the parabollic curve shown in Fig



8

Step 1 For calculation of I_{xx}, use a horizontal strip of width dy as shown

$$I_{xx} = \int_0^{10} y^2 \, dA$$



$$dA = dy \cdot (10 - x)$$

$$x = \sqrt{10y}$$

$$I_{xx} = \int_0^{10} y^2 (10 - \sqrt{10}\sqrt{y})\, dy \Rightarrow \int_0^{10} 10y^2 dy - \sqrt{10}\int_0^{10} y^{5/2} dy$$

$$= 10\left|\frac{y^3}{3}\right|_0^{10} - \sqrt{10}\left[\frac{y^{7/2}}{7/2}\right]_0^{10}$$

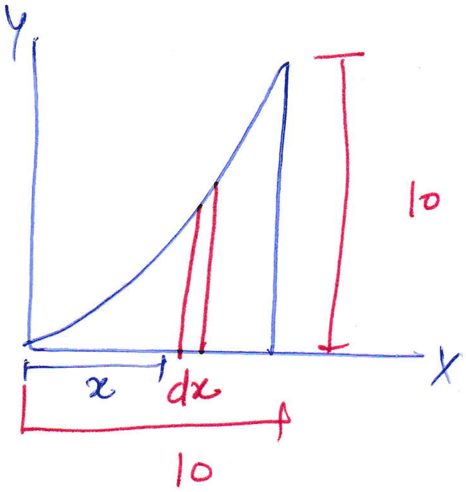
$$\frac{10 \times 10^3}{3} - \frac{\sqrt{10} \times 10^{7/2}}{7/2} = 3333.33 - 2857.147$$

$$= 476.18 \text{ mm}^4$$

Step 2 Calculation of I_{yy}, take a strip along y-axis of breadth dx

I_{yy}

$$I_{yy} = \int_0^{10} x^2 y \, dx$$



$$\int_0^{10} x^2 \cdot \frac{x^2}{10} dx = \left|\frac{x^5}{50}\right|_0^{10}$$

$$= 2000 \text{ mm}^4$$

Step 3 I_{xy} We use an infinitesimal area element dx dy as shown.

$$I_{xy} = \int_0^{10}\int_{y=0}^{y=x^2/10} xy\,dy\,dx$$

$$\int_0^{10} x\left[\int_{y=0}^{y=x^2/10} y\,dy\right]dx$$

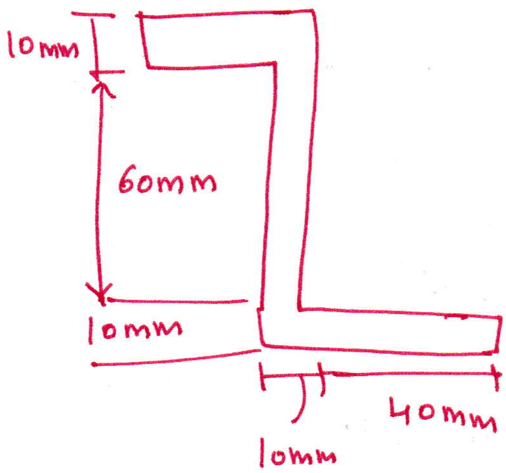
$$\int_0^{10} x\left[\frac{y^2}{2}\right]_{y=0}^{x^3/10} dx = \int_0^{10} x\frac{x^4}{200}\,dx$$

$$= \left.\frac{x^5}{1200}\right|_0^{10} = 833\ mm^4$$

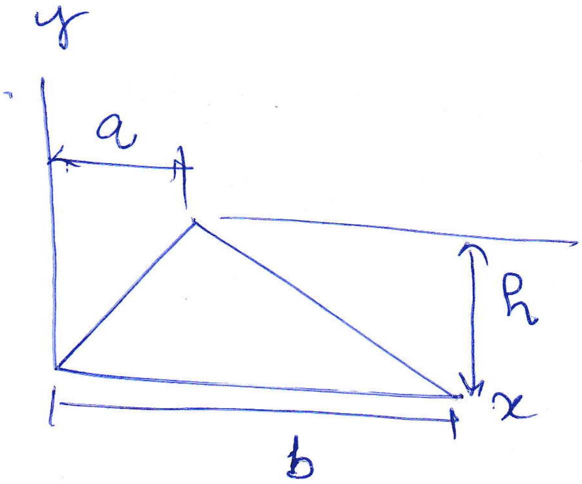
Note: multiple integration involve boundaries requiring some variable limits, in contrast to previous multiple integrations.

Find the centroid of the area of the unequal - leg Z section shown in Fig.

Determine the second moment of area about the centroidal axes parallel to the sides of the Z section. Finally determine the product of inertia about the centeroidal axis.



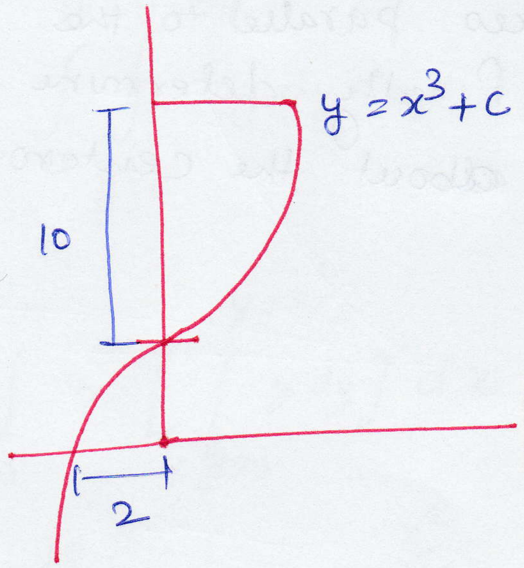
(2)



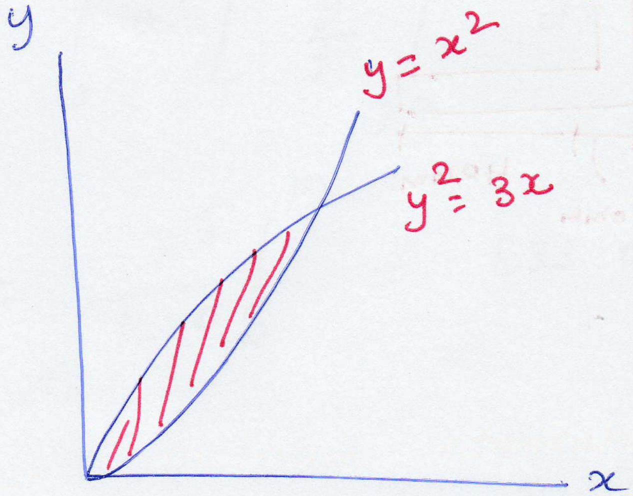
Show that $I_{xx} = bh^3/12$, $I_{yy} = \frac{hb}{12}\left[b^2 + ab + a^2\right]$

$I_{xy} = h^2b/24\,(a+b)$ for a triangle.

(3) Find I_{yy} for the shaded area.
You must determine constant c

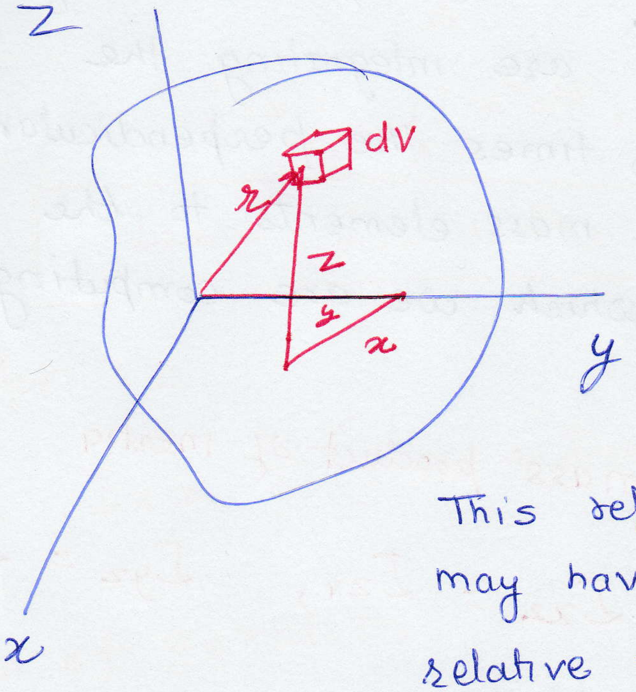


4



Find I_{xx}, I_{yy}, I_{xy} for the shaded area.

Formal Definition of Inertia



A body of mass M and a reference xyz are presented in Fig.

This reference and the body may have any motion whatever relative to each other.

Present discussion Hold for the instantaneous orientation shown at time t.

Consider that body is composed of a continum of particles, each of which has a mass given by ρdV

Now

$$I_{xx} = \iiint_V (y^2+z^2)\,\rho\, dV$$

$$I_{yy} = \iiint_V (x^2+z^2)\,\rho\, dV$$

$$I_{zz} = \iiint_V (x^2+y^2)\,\rho\, dV$$

$$I_{xy} = \iiint_V xy\,\rho\, dV$$

$$I_{xz} = \iiint_V xz\,\rho\, dV$$

$$I_{yz} = \iiint_V yz\,\rho\, dV$$

I_{xx}, I_{yy}, and I_{zz} — mass moments of inertia of body about the x, y, and z axes, respectively.

~~Sam~~ In each case, we are integrating the mass elements ρdV, times the perpendicular distance squared from the mass elements to the coordinate axis about which we are computing the moment of inertia.

I_{xy}, I_{xz}, I_{yz} — mass product of inertia

$I_{xy} = I_{yx}$, $I_{xz} = I_{zx}$, $I_{yz} = I_{zy}$

— 9 component

6 — component are independent.

Note: Depend on the position and inclination of the reference relative to the body.

$$I_{xx} + I_{yy} + I_{zz} = \iiint_V (y^2+z^2)\rho dV + \iiint_V (x^2+z^2)\rho dV + \iiint_V (x^2+y^2)\rho dV$$

$$= 2\iiint_V (x^2+y^2+z^2)\rho dV = \iiint_V 2|r|^2 \rho dV$$

Magnitude of the position vector from the origin to a particle is independent of the inclination of the reference origin. Thus sum of the moments of inertia at a point in space for a given body clearly is an invariant with respect to rotation of axes.

For plane body

$$I_{zz} = \iint_A (x^2+y^2)\rho\, dA$$

$$I_{xx} = \iint_A x^2\, dA, \qquad I_{yy} = \iint y^2\, dA$$

$$\boxed{I_{zz} = I_{xx} + I_{yy}}$$

But for 3D

$$I_{zz} \neq (I_{xx} + I_{yy})$$

Radii of gyration

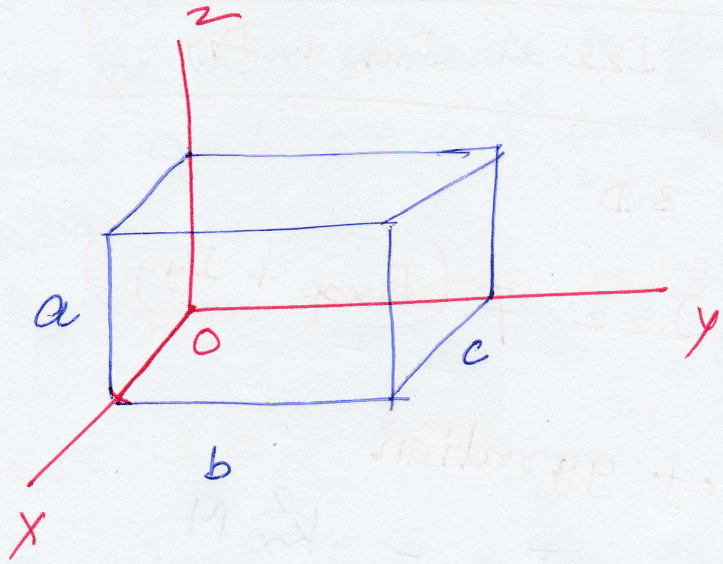
$$I_{xx} = k_x^2 M$$

$$I_{yy} = k_y^2 M$$

$$I_{zz} = k_z^2 M$$

Find the nine component of inertia tensor of a rectangular body of uniform density ρ about point O for a reference xyz coincident with the edges of the block as shown

(B) 4



Find I_{ij} at O

~~Product~~

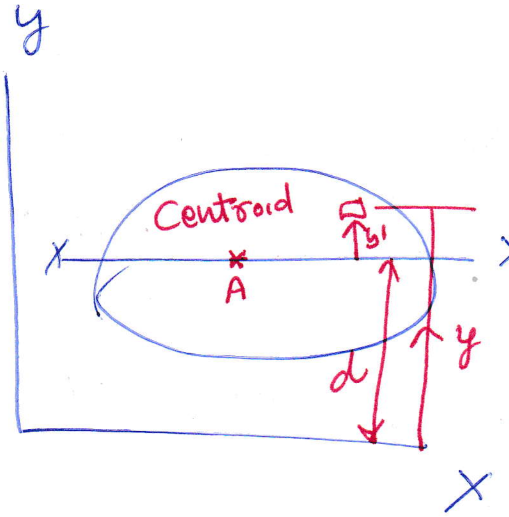
~~For~~ if the area has an axis of symmetry, the product of area for this axes and any axis orthogonal to the axes must be zero.

Note : This does not mean that non symmetric area can not have zero product of area about a set of axes.

Transfer Theorems.

We can find second moments or product of area about any axis in terms of second moments or product of area about a parallel set of axes going through the centroid of the area in question.

x and x' are parallel axis



$$I_{xx} = \int y^2 dA = \int (y'+d)^2 dA$$

$$I_{xx} = \int y'^2 dA + \int d^2 dA + \overbrace{\int 2y'd\,dA}^{0}$$

x' - going through the centroid of the area.

$$\boxed{I_{xx} = I'_{xx} + d^2 A}$$