**Question 1 (see Fig. 1)** Two $2 \times 4$ m plywood panels, each of weight 60 N, are nailed together. The panels are supported by ball-and-socket joints at $A$ and $F$ and by the wire $BH$. It is required that the tension in the wire be minimum. Appropriate location of $H(x,y)$ in the $xy$ plane and the corresponding minimum tension $T_{\text{min}}$ are to be determined.

(a) Draw the free-body diagram of the panels (combined) and derive the equation(s) of equilibrium, in which the only unknown is the force $T$ (vector). [10 marks]

(b) Using vector identities or otherwise, determine $T_{\text{min}}$ and corresponding $(x,y)$. [10 marks]
SOLUTION:

Part (a)

\[ \mathbf{AF} = 4\hat{i} - 2\hat{j} - 4\hat{k} \]

\[ AF = \sqrt{4^2 + 2^2 + 4^2} = 6 \text{ m} \]

\[ \lambda_{AF} = \frac{AF}{AF} = \frac{1}{3}(2\hat{i} - \hat{j} - 2\hat{k}) \]

\[ \mathbf{r}_{G_1/A} = 2\hat{i} - \hat{j} \]

\[ \mathbf{r}_{G_2/A} = 4\hat{i} - \hat{j} - 2\hat{k} \]

\[ \mathbf{r}_{B/A} = 4\hat{i} \]

\[ M_{AF}^{G_1} = \lambda_{AF} \cdot (\mathbf{r}_{G_1/A} \times \mathbf{G}_1) = \frac{1}{3} \begin{vmatrix} 2 & -1 & -2 \\ 2 & -1 & 0 \\ 0 & -60 & 0 \end{vmatrix} = \frac{1}{3} \times 60 \times (0 + 4) = 80 \text{ N m} \]

\[ M_{AF}^{G_2} = \lambda_{AF} \cdot (\mathbf{r}_{G_2/A} \times \mathbf{G}_2) = \frac{1}{3} \begin{vmatrix} 2 & -1 & -2 \\ 4 & -1 & -2 \\ 0 & -60 & 0 \end{vmatrix} = \frac{1}{3} \times 60 \times (-4 + 8) = \frac{1}{3} \times 60 \times 4 = 80 \text{ N m} \]
\[ M_{AF} = \lambda_{AF} \cdot (\mathbf{r}_{B/A} \times \mathbf{T}) \]

\[ \sum M_{AF} = 0 \]

\[ 80 + 80 + \lambda_{AF} \cdot (\mathbf{r}_{B/A} \times \mathbf{T}) = 0 \]

\[ \lambda_{AF} \cdot (\mathbf{r}_{B/A} \times \mathbf{T}) = -160 \hspace{1cm} (1) \]

**Part (b)**

Using the property of scalar triple product, Eq. (1) can be rewritten as

\[ \mathbf{T} \cdot (\mathbf{r}_{B/A} \times \lambda_{AF}) = -160 \]

Thus, the projection of \( \mathbf{T} \) on \( (\lambda_{AF} \times \mathbf{r}_{B/A}) \) is constant.

Hence, \( \mathbf{T}_{\text{min}} \) is parallel to \( \lambda_{AF} \times \mathbf{r}_{B/A} \),

\[ \lambda_{AF} \times \mathbf{r}_{B/A} = \frac{1}{3} (2\mathbf{i} - \mathbf{j} - 2\mathbf{k}) \times 4\mathbf{i} \]

\[ = \frac{1}{3} \begin{vmatrix} i & j & k \\ 2 & -1 & -2 \\ 4 & 0 & 0 \end{vmatrix} = \frac{1}{3} (-8\mathbf{j} + 4\mathbf{k}) \]

Magnitude, \( |\lambda_{AF} \times \mathbf{r}_{B/A}| = \frac{1}{3} \sqrt{8^2 + 4^2} = \frac{1}{3} \sqrt{80} = \frac{4}{3} \sqrt{5} \)

Corresponding unit vector \( = \frac{\lambda_{AF} \times \mathbf{r}_{B/A}}{|\lambda_{AF} \times \mathbf{r}_{B/A}|} = \frac{\frac{1}{3} (-8\mathbf{j} + 4\mathbf{k})}{\frac{4}{3} \sqrt{5}} \]

\[ = \frac{1}{\sqrt{5}} (-2\mathbf{j} + \mathbf{k}) \]

\[ \Rightarrow \mathbf{T}_{\text{min}} = \frac{T}{\sqrt{5}} (-2\mathbf{j} + \mathbf{k}) \hspace{1cm} (2) \]

Eq. (1):

\[ \frac{T}{\sqrt{5}} (-2\mathbf{j} + \mathbf{k}) \cdot \left[ \frac{1}{3} (2\mathbf{i} - \mathbf{j} - 2\mathbf{k}) \times 4\mathbf{i} \right] = -160 \]

\[ \Rightarrow \frac{T}{\sqrt{5}} (-2\mathbf{j} + \mathbf{k}) \cdot \frac{1}{3} (-8\mathbf{j} + 4\mathbf{k}) = -160 \]

\[ \Rightarrow \frac{T}{3\sqrt{5}} (16 + 4) = -160 \]

\[ \Rightarrow T = -\frac{3\sqrt{5}(160)}{20} = 24\sqrt{5} = 53.67 \text{ N} \]
Eq. (2):

\[ T_{\text{min}} = \frac{T}{\sqrt{5}} (-2\hat{j} + \hat{k}) = 24\sqrt{5}(-2\hat{j} + \hat{k}) \frac{1}{\sqrt{5}} \]

\[ \Rightarrow T_{\text{min}} = (-48 \text{ N})\hat{j} + (24 \text{ N})\hat{k} \]

Since \( T_{\text{min}} \) has no \( \hat{i} \) component, wire \( BH \) is parallel to the \( y-z \) plane.

Hence, \( x = 4 \text{ m} \).

From the diagram, \( y = 8.00 \text{ m} \).

So, the answers are the following.

\( x = 4.00 \text{ m} \)

\( y = 8.00 \text{ m} \)

\( T_{\text{min}} = 53.7 \text{ N} \)
Alternate Solution for Part (b)

Eq. (1):
\[ \lambda_{AF} \cdot (r_{B/A} \times T) = -160 \]

\[ T = T\hat{\lambda}_{BH} \]

\[ \lambda_{BH} = \frac{BH}{BH} = \frac{(x - 4)i + yj - 4k}{[(x - 4)^2 + y^2 + 16]^{1/2}} \]

\[ \begin{vmatrix} 2 & -1 & -2 & T \\ 4 & 0 & 0 \\ x - 4 & y & -4 \end{vmatrix} \frac{1}{[(x - 4)^2 + y^2 + 16]^{1/2}} = -160 \]

\[ \Rightarrow -\frac{4}{3} (4 + 2y) \frac{T}{[(x - 4)^2 + y^2 + 16]^{1/2}} = -160 \]

\[ \Rightarrow T = \frac{60[(x - 4)^2 + y^2 + 16]^{1/2}}{(2 + y)} \]

\[ \Rightarrow T^2 = \frac{3600[(x - 4)^2 + y^2 + 16]}{(y + 2)^2} \]

\[ = 3600 f(x, y) \]

where \( f(x, y) = \frac{[(x - 4)^2 + y^2 + 16]}{(y + 2)^2} \)

\[ f_x(x, y) = \frac{2(x - 4)}{(y + 2)^2} \]

\[ f_y(x, y) = 0 \]

\[ \Rightarrow \frac{2(x - 4)}{(y + 2)^2} = 0 \]

\[ \Rightarrow x - 4 = 0 \]

\[ \Rightarrow x = 4 \]
\[ f_y(x, y) = \frac{2y}{(y + 2)^2} - \frac{2[(x - 4)^2 + y^2 + 16]}{(y + 2)^3} \]
\[ = \frac{2}{(y + 2)^3} [y(y + 2) - (x - 4)^2 + y^2 + 16] \]
\[ = \frac{2}{(y + 2)^3} [2y - (x - 4)^2 - 16] \]

Putting \( f_y(x, y) = 0 \)
\[ \Rightarrow \frac{2}{(y + 2)^3} [2y - (x - 4)^2 - 16] = 0 \]
\[ \Rightarrow 2y - (x - 4)^2 - 16 = 0 \]
Putting \( x = 4 \) gives
\[ y = 8 \]

Thus, the critical point is \((4, 8)\)

Now, \( T = \frac{60[(x-4)^2 + y^2 + 16]^{1/2}}{y+2} \)
\[ \therefore T_{\text{min}} = \frac{60[(4-4)^2 + 8^2 + 16]^{1/2}}{8 + 2} \]
\[ \Rightarrow T_{\text{min}} = 53.76 \text{ N} \]

So, the answers are the following.
\[ x = 4.00 \text{ m} \]
\[ y = 8.00 \text{ m} \]
\[ T_{\text{min}} = 53.7 \text{ N} \]
**Confirmatory Test:**

Now, \( f_{xx}(x, y) = \frac{2}{(y+2)^2} \)

Thus, \( f_{xx}(4,8) = \frac{2}{10^2} = 0.02 \)

And \( f_{yy}(x, y) = -\frac{6}{(y+2)^4} \{ 2y - (x - 4)^2 - 16 \} + \frac{2}{(y+2)^3} \)

\( f_{yy}(4,8) = 0 + \frac{4}{10^3} = 0.004 \)

Again, \( f_{xy}(x, y) = -\frac{4(x-4)}{(y+2)^3} \)

So, \( f_{xy}(4,8) = 0 \)

Now, \( D = [f_{xx} \cdot f_{yy} - f_{xy}^2]_{(4,8)} \)

\( = 0.02 \times 0.004 - 0 \)

\( \Rightarrow D = 8 \times 10^{-5} > 0 \)

Also, \( f_{xx}(4,8) > 0 \)

Thus, \( T = T_{\text{min}} \) at \( (x, y) = (4,8) \)
Question-2:
Find out the support reactions and the forces in the members DF, CE & CF of the truss shown in Figure-1, with proper labels (tension or compression) using the methods of sections?

Solution:
From the entire structure, we find the reactions at A.

\[ \sum F_x: \quad A_x = 0 \]

\[ \sum M_i: \quad (200kN)(5m) + (200kN)(10m) + (200kN)(15m) + (200kN)(20m) \]
\[ - A_y(20m) = 0 \Rightarrow A_y = 500kN \]

Now we cut through DF, CF and CE and use the left section.

\[ \sum M_c: \quad (200kN)(5m) - A_y(5m) + A_x(3m) - F_{DF}(4m) = 0 \]

\[ F_{DF} = -375 \text{ kN} \]

\[ \sum M_F: \quad (200kN)(10m) + (200kN)(5m) - A_y(10m) + A_x(7m) \]
\[ + \frac{5}{\sqrt{26}} F_{CE}(4m) - \frac{1}{\sqrt{26}} F_{CE}(5m) = 0 \Rightarrow F_{CE} = 680kN \]
\[ \sum F_x: \ Ax + F_{DF} + \frac{5}{\sqrt{26}} F_{CE} + \frac{5}{\sqrt{41}} F_{CF} = 0 \]

\[ F_{CF} = 374 \text{ kN} \]
**Question 3:**
For the beam ABC shown in Figure. 3 (The member DB is a two-force member), answer the following questions:

a) Mention the sign convention for shear force and bending moment
b) Find out the expressions of the shear force and bending moment as a function of distance \( x \) considered from the support A

c) Draw the shear force and the bending moment diagrams. Find out the location of maximum bending moment.

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**Figure 3**

**SOLUTION**

a.) Sign Conventions

**Shear sign convention**
BMD sign convention

Positive

Sagging positive and Hogging negative

b.) SFD Functions:
1) At $x = 0$

SHEAR FORCE $= -11 \, kN$

2) For $0 < x \leq 2$:

SHEAR FORCE $= -11 - 4x \, kN$

3) For $2 < x \leq 4$:

SHEAR FORCE $= -15 \, kN$

BMD Functions:

1) At $x = 0$

BENDING MOMENT $= 0 \, kNm$

2) For $0 < x \leq 2$:

BENDING MOMENT $= -11x - 2x^2 \, kNm$

3) For $2 < x \leq 4$ (x measured from the right end):

BENDING MOMENT $= -15x \, kNm$
c.) To find maximum bending moment, we have to differentiate the BMD Function wrt x, and equate it to zero –

\[
\frac{d}{dx} \left(-11x - 2x^2\right) = 0 \quad \Rightarrow \quad -11 - 4x = 0 \quad \Rightarrow \quad x = -2.75 \, m
\]

This means that the maxima lies outside the region of \(0 < x \leq 2\), and hence, the curve is steeply falling in this region. Thus, the maximum bending moment (absolute maximum, i.e. minimum in this context) will occur at the end of the region (\(x = 2\))

Maximum Bending Moment = \(-15 \times 2\) = \(-30 \, kNm\)
Case I: If block B moves to the right:

Let \( P \) be the applied force on B. Free body diagrams of blocks are

Equilibrium of block A:

\[
F_y = 0 \Rightarrow -3000 + N_1 \cos 30 - N_2 \sin 30 = 0
\]

\[\text{or } N_1 (\cos 30 - N_2 \sin 30) = 3000\]

\[N_1 = \frac{2000}{\cos 30 - 0.12 \sin 30} = \frac{3000}{0.806} = 3721.97 \text{ N}\]

Equilibrium of block B:

\[
F_y = 0 \Rightarrow -N_1 \cos 30 + N_2 \sin 30 + 2000 + N_2 = 0
\]

\[\text{or } N_2 = 2000 + N_1 (\cos 30 - N_2 \sin 30)\]

\[= 2000 + 3721.97 \times 0.806\]

\[N_2 = 5000 \text{ N}\]

\[
F_x = 0 \Rightarrow P - M_s N_2 - N_3 N_1 \cos 30 - N_1 \sin 30 = 0
\]

\[\text{or } P = 0.12 \times 5000 + N_1 (0.12 \cos 30 + \sin 30)\]

\[P = 2847.78 \text{ N}\]

The given force 400 N is less than the required force to move block B to the right. Hence block B will not move to the right.
Case II: If block B moves to the left:

Let applied force on block B be $P_1$. Free body diagrams of blocks are

**Equilibrium of block A:**

$F_y = 0 \Rightarrow -3000 + N_1 \cos 30 + \mu_s N_1 \sin 30 = 0$

or $N_1 = \frac{-3000}{\cos 30 + 0.12 \times \sin 30} = \frac{-3000}{0.926} = 3239.65 \text{N}$

**Equilibrium of block B:**

$F_y = 0 \Rightarrow N_2 - 2000 - N_1 \cos 30 - \mu_s N_1 \sin 30 = 0$

or $N_2 = 2000 + N_1 \kappa (\cos 30 + \mu_s \sin 30)$

$N_2 = 5000 \text{N}$

$F_x = 0 \Rightarrow P_1 + \mu_s N_2 + \mu_s N_1 \cos 30 = N_1 \sin 30 = 0$

or $P_1 = N_1 (\sin 30 - \mu_s \cos 30) - \mu_s N_2$

$P_1 = 3239.65 \times 0.396 - 0.12 \times 5000$

$P_1 = 683.15 \text{N}$

As the applied force (given) 400N is less than the required force to move block B to the left, the block B will move to the left.

The blocks are not in equilibrium.
Consider pulley B.

We have

$$\beta_B = 180 + 2 \times 8.63$$

or

$$\beta_B = 197.26^\circ$$

For impending friction, we have

$$\frac{T_1}{T_2} = e^{u_s \beta_B} = e^{(u_s \times \frac{197.26}{180} \times T)} 
\Rightarrow \frac{T_1}{T_2} = e^{3.44 N_s} \quad (1)$$

We have moment relation

$$ (T_1 - T_2) \cdot 8 = 750 \Rightarrow T_1 - T_2 = 93.75 \quad (2)$$

$$Fx = 0 \text{ for the pulley gives}$$

$$ (T_1 + T_2) \cos(18.63) = 3000$$

or

$$T_1 + T_2 = 3034.255 \quad (3)$$

Addition of equations (2) and (3) gives

$$2T_1 = 3971.855 \Rightarrow T_1 = 1985.93 N$$

Substituting $T_1$ in equation (2) and solving for $T_2$, we get

$$T_2 = 1985.93 - 93.75 = 1048.43$$

Substituting $T_1$ and $T_2$ in eqn (1) we get

$$\frac{1985.93}{1048.43} = e^{3.44 N_s}$$

or

$$N_s = \frac{1}{3.44} \ln \left(\frac{1985.93}{1048.43}\right) = 0.186$$

$$N_s = 0.186$$
Consider pulley A

\[
T_1 = 3000 \text{ N}
\]

\[
\beta_A = 180 - 2 \times 8.63 \]

\[
\theta_A = 162.74^\circ
\]

We have for impending slippage,

\[
\frac{T_1}{T_2} = e^{\left(\frac{162.74 \times 1}{180}\right)} M_3 = e^{2.84 M_3}
\]

\(T_1\) and \(T_2\) will be the same as for the case of pulley B. Substituting \(T_1\) and \(T_2\) in the above expression,

\[
\frac{1985.93}{1048.43} = e^{2.84 M_3}
\]

\[
M_3 = \frac{1}{2.84} \ln\left(\frac{1985.93}{1048.43}\right)
\]

\[
M_3 = 0.225
\]

Minimum static coefficient of friction for applied force and moment without slippage to run the system is the maximum of the two values obtained for pulley A and B.

\[
M_3 = 0.225 \text{ for both the pulley}
\]
Question 6 (Solution)

When link is located in the arbitrary position \( \theta \),

- Increase in P.E due to stretching of spring = \( V_e = \frac{1}{2} ks^2 \)
- Decrease in P.E due to weight = \( V_g = W_y \cdot y \)

Total P.E = \( V_e + V_g = \frac{1}{2} ks^2 + W_y = (V) \)

\[ L = s + \text{coast} \quad \text{(fig 2)} \]

\[ \Rightarrow a = L (1 - \cos \theta) \]

\[ \Rightarrow y = \left( \frac{L}{2} \right) \cos \theta. \]

\[ \therefore V = \frac{1}{2} kl^2 (1 - \cos \theta)^2 + W \left( \frac{L}{2} \cos \theta \right) \]

Equilibrium position: \[ \frac{dV}{d\theta} = \frac{kl^2 (1 - \cos \theta)}{2} \sin \theta - \frac{Wl \sin \theta}{2} = 0 \]

\[ \therefore L \left[ kl (1 - \cos \theta) - \frac{W}{2} \right] \sin \theta = 0 \]

Show this equation is satisfied provided

\[ \sin \theta = 0 \quad \Rightarrow \theta = 90^\circ. \]

\[ \text{or} \: \theta = \cot^{-1} \left( 1 - \frac{W}{2kl} \right) = \cot^{-1} \left[ 1 - \frac{10 \cdot 9.81}{2 \cdot 200 \cdot 0.6} \right] \approx 53.8^\circ. \]
Stability

\[
\frac{d^2V}{d\theta^2} = kl^2 (1 - \cos \theta) \cos \theta + kl^2 \sin \theta \sin \theta = -\frac{wl}{2} \cos \theta
\]

\[
= kl^2 (\cos \theta - \cos 2\theta) - \frac{wl}{2} \cos \theta
\]

Substituting values for constants, with \( \theta = 0^\circ \) and \( \theta = 53.8^\circ \):

\[
\left. \frac{d^2V}{d\theta^2} \right|_{\theta = 0^\circ} = 200 (0.6)^2 (\cos 0^\circ - \cos 0^\circ) - \frac{10 (9.81) (0.6)}{2} \cos 0^\circ
\]

\[= -29.4 \text{ which is } < 0 \text{ (unstable equilibrium at } \theta = 0^\circ)\]

\[
\left. \frac{d^2V}{d\theta^2} \right|_{\theta = 53.8^\circ} = 200 (0.6)^2 (\cos 53.8^\circ - \cos 107.6^\circ) - \frac{10 (9.81) (0.6)}{2} \cos 53.8^\circ
\]

\[= 46.9 > 0 \text{ (stable equilibrium at } \theta = 53.8^\circ)\]