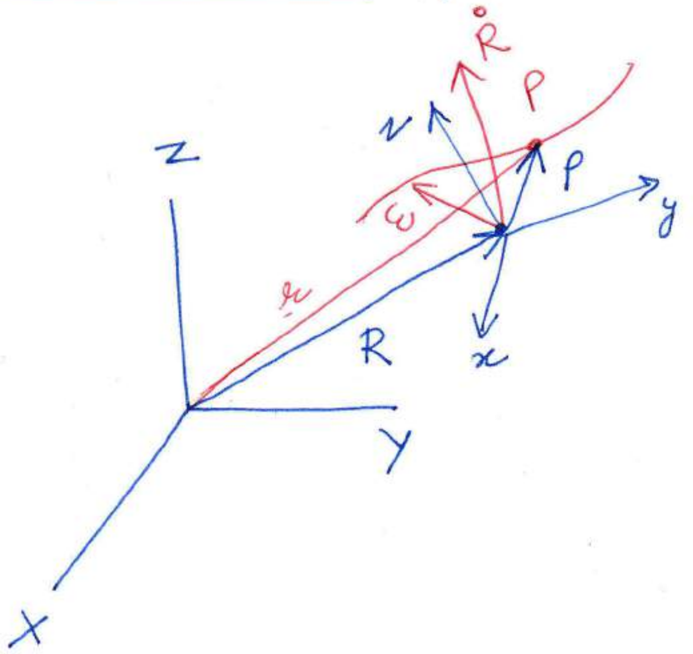
General Relationship



Time derivative of a vector 'A' fixed in a reference xyz moving arbitrarily relative to XYZ. Our conclusions were

$$\left(\frac{dA}{dt}\right)_{xyz} = 0$$

$$\left(\frac{dA}{dt}\right)_{XYZ} = \omega \times A$$

Extend these consideration to include time derivative of vector A which is not necessarily fixed in reference xyz.

For this purpose, a moving particle P with a position vector ρ in reference xyz is considered.

$\dot{R}$ = Translation velocity

ω = ~~rotational~~ angular velocity

Now ρ can be expressed interms of components parallel to xyz ~~refrencen~~ reference

$$\rho = x\hat{i} + y\hat{j} + z\hat{k}$$

Where $i, \hat{j}, \hat{k}$ are unit vectors for reference xyz

$$\left(\frac{d\rho}{dt}\right)_{xyz} = \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}$$

Now Take derivative of ρ with respect to time for the XYZ reference.

$$\left(\frac{d\rho}{dt}\right)_{XYZ} = (\dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}) + (x\dot{\hat{i}} + y\dot{\hat{j}} + z\dot{\hat{k}})$$

$$\frac{di}{dt} = \omega \times i, \quad \frac{dj}{dt} = \omega \times \hat{j}, \quad \frac{dk}{dt} = \omega \times \hat{k}$$

$$\left(\frac{d\rho}{dt}\right)_{XYZ} = \left(\frac{d\rho}{dt}\right)_{xyz} + x(\omega \times \hat{i}) + y(\omega \times \hat{j}) + z(\omega \times \hat{k})$$

$$= \left(\frac{d\rho}{dt}\right)_{xyz} + \omega \times (x\hat{i} + y\hat{j} + z\hat{k})$$

($x\hat{i} + y\hat{j} + z\hat{k} \Rightarrow \rho$)

$$\left(\frac{d\rho}{dt}\right)_{XYZ} = \left(\frac{d\rho}{dt}\right)_{xyz} + \omega \times \rho$$

AB

Relationship Between Velocities of a particle for different references.

1 Velocities of particle P relative to the XYZ and xyz reference are, respectively

$$V_{XYZ} = \left(\frac{dr}{dt}\right)_{XYZ}, \quad V_{xyz} = \left(\frac{d\rho}{dt}\right)_{xyz}$$

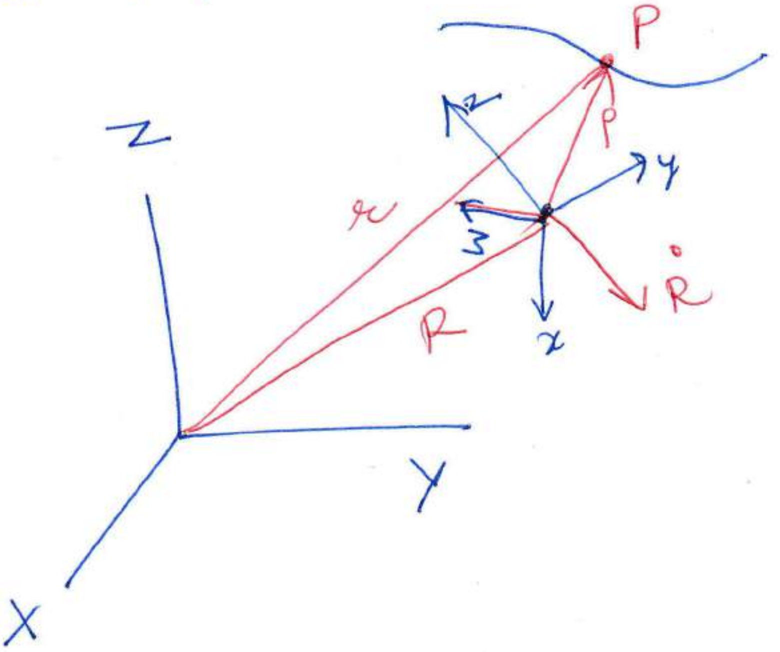
$$r = R + \rho$$

$$\left(\frac{dr}{dt}\right)_{XYZ} = V_{XYZ} = \left(\frac{dR}{dt}\right)_{XYZ} + \left(\frac{d\rho}{dt}\right)_{XYZ}$$

$$V_{XYZ} = \dot{R} + \left(\frac{d\rho}{dt}\right)_{xyz} + \omega \times \rho$$

$$\boxed{V_{XYZ} = \dot{R} + V_{xyz} + \omega \times \rho}$$

Acceleration of a particle for different References.



$$a_{XYZ} = \left[\frac{d}{dt}(V_{XYZ})\right]_{XYZ} = \left(\frac{d^2 r}{dt^2}\right)_{XYZ}$$

$$a_{xyz} = \left[\frac{d}{dt}(V_{xyz})\right]_{xyz} = \left(\frac{d^2 \rho}{dt^2}\right)_{xyz}$$

$$\boxed{V_{XYZ} = V_{xyz} + \dot{R} + \omega \times \rho} \quad (1)$$

Differentiating w.r.t. time for XYZ

$$\left(\frac{d V_{XYZ}}{dt}\right)_{XYZ} = a_{XYZ} = \left(\frac{d V_{xyz}}{dt}\right)_{XYZ} + \ddot{R} + \frac{d}{dt}(\omega \times \rho)_{XYZ}$$

$$a_{XYZ} = \left(\frac{d V_{xyz}}{dt}\right)_{XYZ} + \ddot{R} + \omega \times \left(\frac{d\rho}{dt}\right)_{XYZ} + \left(\frac{d\omega}{dt}\right)_{XYZ} \times \rho \quad (2)$$

Now

$$\left(\frac{dV_{xyz}}{dt}\right)_{XYZ} = \left(\frac{dV_{xyz}}{dt}\right)_{xyz} + \omega \times V_{xyz}$$

$$\left(\frac{d\rho}{dt}\right)_{XYZ} = \left(\frac{d\rho}{dt}\right)_{xyz} + \omega \times \rho$$

Substitute the above relations in eq. (2)

$$a_{XYZ} = a_{xyz} + \omega \times V_{xyz} + \ddot{R} + \omega \times V_{xyz} + \omega \times (\omega \times \rho) + \dot{\omega} \times \rho$$

$$a_{XYZ} = a_{xyz} + 2\omega \times V_{xyz} + \ddot{R} + \dot{\omega} \times \rho + \omega \times (\omega \times \rho)$$

ω = angular velocity of xyz relative to XYZ

$\dot{\omega}$ = angular acceleration of the xyz reference relative to XYZ

$2(\omega \times V_{xyz})$ = Coriolis acceleration vector

Example 15.11

A stationary truck is carrying a cockpit for a worker who repairs overhead fixtures. At the instant shown in Fig. 15.30, the base D is rotating at angular speed ω_2 of .1 rad/sec with $\dot{\omega}_2 = .2$ rad/sec^2 relative to the truck. Arm AB is rotating at angular speed ω_1 of .2 rad/sec with $\dot{\omega}_1 = .8$ rad/sec^2 relative to DA. Cockpit C is rotating relative to AB so as to always keep the man upright. What are the velocity and acceleration vectors of the man relative to the ground if $\alpha = 45°$ and $\beta = 30°$ at the instant of interest? Take $DA = 13$ m.

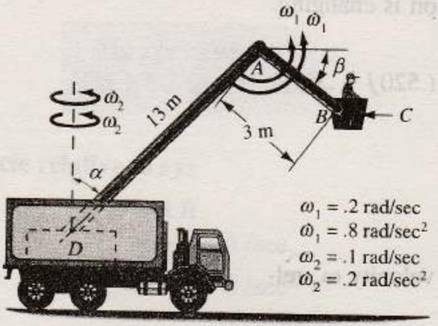


Figure 15.30. Truck with moving cockpit.

Because of the rotation of the cockpit C relative to arm AB to keep the man vertical, clearly, each particle in that body including the man has the same motion as point B of arm AB. Therefore, we shall concentrate our attention on this point.

Fix xyz to arm DA.
Fix XYZ to truck.

This situation is shown in Fig. 15.31.

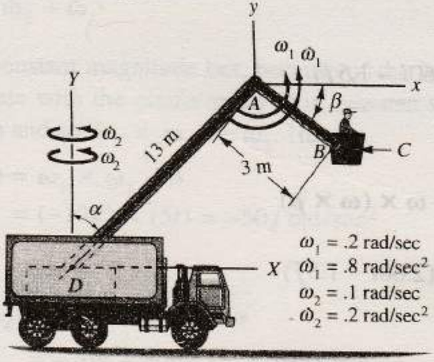


Figure 15.31. xyz fixed to DA; XYZ fixed to truck.

A. Motion of B relative to xyz

$$\boldsymbol{\rho} = 3(\cos\beta\boldsymbol{i} - \sin\beta\boldsymbol{j}) = 2.60\boldsymbol{i} - 1.5\boldsymbol{j}\text{ m}$$

Example 15.11 (Continued)

Since $\boldsymbol{\rho}$ is fixed in AB, which has angular velocity $\boldsymbol{\omega}_1$ relative to xyz, we have

$$\begin{aligned} \boldsymbol{V}_{xyz} &= \boldsymbol{\omega}_1 \times \boldsymbol{\rho} = (.2\boldsymbol{k}) \times (2.60\boldsymbol{i} - 1.5\boldsymbol{j}) \\ &= .520\boldsymbol{j} + .3\boldsymbol{i} \text{ m/sec} \\ \boldsymbol{a}_{xyz} &= \left(\frac{d\boldsymbol{\omega}_1}{dt}\right)_{xyz} \times \boldsymbol{\rho} + \boldsymbol{\omega}_1 \times \left(\frac{d\boldsymbol{\rho}}{dt}\right)_{xyz} \end{aligned}$$

As seen from xyz, only the value of $\boldsymbol{\omega}_1$ and not its direction is changing. Also note that $(d\boldsymbol{\rho}/dt)_{xyz} = \boldsymbol{V}_{xyz}$. Hence,

$$\begin{aligned} \boldsymbol{a}_{xyz} &= (.8\boldsymbol{k}) \times (2.60\boldsymbol{i} - 1.5\boldsymbol{j}) + (.2\boldsymbol{k}) \times (.520\boldsymbol{j} + .3\boldsymbol{i}) \\ &= 1.09\boldsymbol{i} + 2.14\boldsymbol{j} \text{ m/sec}^2 \end{aligned}$$

B. Motion of xyz relative to XYZ

$$\boldsymbol{R} = 13(.707\boldsymbol{i} + .707\boldsymbol{j}) = 9.19\boldsymbol{i} + 9.19\boldsymbol{j} \text{ m}$$

Since $\boldsymbol{R}$ is fixed in DA, and since DA rotates with angular velocity $\boldsymbol{\omega}_2$ relative to XYZ, we have

$$\begin{aligned} \dot{\boldsymbol{R}} &= \boldsymbol{\omega}_2 \times \boldsymbol{R} = (.1\boldsymbol{j}) \times (9.19\boldsymbol{i} + 9.19\boldsymbol{j}) \\ &= -.919\boldsymbol{k} \text{ m/sec} \\ \ddot{\boldsymbol{R}} &= \dot{\boldsymbol{\omega}}_2 \times \boldsymbol{R} + \boldsymbol{\omega}_2 \times \dot{\boldsymbol{R}} \\ &= (.2\boldsymbol{j}) \times (9.19\boldsymbol{i} + 9.19\boldsymbol{j}) + (.1\boldsymbol{j}) \times (-.919\boldsymbol{k}) \\ &= -1.838\boldsymbol{k} - .0919\boldsymbol{i} \text{ m/sec}^2 \\ \boldsymbol{\omega} &= \boldsymbol{\omega}_2 = .1\boldsymbol{j} \text{ rad/sec} \\ \dot{\boldsymbol{\omega}} &= \dot{\boldsymbol{\omega}}_2 = .2\boldsymbol{j} \text{ rad/sec}^2 \end{aligned}$$

Hence,

$$\begin{aligned} \boldsymbol{V}_{XYZ} &= \boldsymbol{V}_{xyz} + \dot{\boldsymbol{R}} + \boldsymbol{\omega} \times \boldsymbol{\rho} \\ &= .520\boldsymbol{j} + .3\boldsymbol{i} - .919\boldsymbol{k} + (.1\boldsymbol{j}) \times (2.60\boldsymbol{i} - 1.5\boldsymbol{j}) \end{aligned}$$

$$\boldsymbol{V}_{XYZ} = .3\boldsymbol{i} + .520\boldsymbol{j} - 1.179\boldsymbol{k} \text{ m/sec}$$

$$\begin{aligned} \boldsymbol{a}_{XYZ} &= \boldsymbol{a}_{xyz} + \ddot{\boldsymbol{R}} + 2\boldsymbol{\omega} \times \boldsymbol{V}_{xyz} + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) \\ &= 1.09\boldsymbol{i} + 2.14\boldsymbol{j} - 1.838\boldsymbol{k} - .0919\boldsymbol{i} \\ &\quad + 2(.1\boldsymbol{j}) \times (.520\boldsymbol{j} + .3\boldsymbol{i}) + (.2\boldsymbol{j}) \times (2.60\boldsymbol{i} - 1.5\boldsymbol{j}) \\ &\quad + (.1\boldsymbol{j}) \times [(.1\boldsymbol{j}) \times (2.60\boldsymbol{i} - 1.5\boldsymbol{j})] \end{aligned}$$

$$\boldsymbol{a}_{XYZ} = .978\boldsymbol{i} + 2.14\boldsymbol{j} - 2.42\boldsymbol{k} \text{ m/sec}^2$$

Notice that the essential aspects of the analysis come in the consideration of parts A and B of the problem, while the remaining portion involves direct substitution and vector algebraic operations.

Example 15.12

A wheel rotates with an angular speed ω_2 of 5 rad/sec on a platform which rotates with a speed ω_1 of 10 rad/sec relative to the ground as shown in Fig. 15.32. A valve gate *A* moves down the spoke of the wheel, and when the spoke is vertical the valve gate has a speed of 20 ft/sec, an acceleration of 10 ft/sec^2 along the spoke, and is 1 ft from the shaft centerline of the wheel. Compute the velocity and acceleration of the valve gate relative to the ground at this instant.

Fix xyz to wheel.
Fix XYZ to ground.

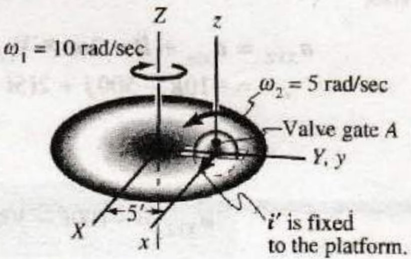


Figure 15.32. *xyz* fixed to wheel; *XYZ* fixed to ground.

A. Motion of particle relative to *xyz*

$$\rho = k \text{ ft}$$
$$V_{xyz} = -20k \text{ ft/sec}$$
$$a_{xyz} = -10k \text{ ft/sec}^2$$

B. Motion of *xyz* relative to *XYZ*

$$R = 5j \text{ ft}$$

Since R is fixed to the platform:

$$\dot{R} = \omega_1 \times R = (-10k) \times (5j) = 50i \text{ ft/sec}$$
$$\ddot{R} = \dot{\omega}_1 \times R + \omega_1 \times \dot{R}$$
$$= 0 + (-10k) \times (50i) = -500j \text{ ft/sec}^2$$
$$\omega = \omega_2 + \omega_1 = 5i - 10k \text{ rad/sec}$$
$$\dot{\omega} = \dot{\omega}_2 + \dot{\omega}_1$$

Note that ω_2 is of constant magnitude but, because of the bearings of the wheel, ω_2 must rotate with the platform. In short, we can say that ω_2 is *fixed* to the platform and so $\dot{\omega}_2 = \omega_1 \times \omega_2$. Hence,

$$\dot{\omega} = \omega_1 \times \omega_2 + 0$$
$$= (-10k) \times (5i) = -50j \text{ rad/sec}^2$$

We then have

$$V_{XYZ} = V_{xyz} + \dot{R} + \omega \times \rho$$
$$= -20k + 50i + (5i - 10k) \times k$$

$$V_{XYZ} = 50i - 5j - 20k \text{ ft/sec}$$

Also,

$$a_{XYZ} = a_{xyz} + \ddot{R} + 2\omega \times V_{xyz} + \dot{\omega} \times \rho + \omega \times (\omega \times \rho)$$
$$= -10k - 500j + 2(5i - 10k) \times (-20k) + (-50j) \times k + (5i - 10k) \times [(5i - 10k) \times k]$$

$$a_{XYZ} = -100i - 300j - 35k \text{ ft/sec}^2$$

Example 15.13

In Example 15.12, the wheel accelerates at the instant under discussion with $\dot{\omega}_2 = 5 \text{ rad/sec}^2$, and the platform accelerates with $\dot{\omega}_1 = 10 \text{ rad/sec}^2$ (see Fig. 15.32). Find the velocity and acceleration of the valve gate A.

If we review the contents of parts A and B of Example 15.12, it will be clear that only $\ddot{\boldsymbol{R}}$ and $\dot{\boldsymbol{\omega}}$ are affected by the fact that $\dot{\omega}_1 = 10 \text{ rad/sec}^2$ and $\dot{\omega}_2 = 5 \text{ rad/sec}^2$. In this regard, consider $\boldsymbol{\omega}_2$. It is no longer of constant value and cannot be considered as *fixed* in the platform. However we can express $\boldsymbol{\omega}_2$ as $\omega_2 \boldsymbol{i}'$ *at all times*, wherein $\boldsymbol{i}'$ is *fixed* in the platform as shown in Fig. 15.32. Thus, we can say for $\boldsymbol{\omega}$:

$$\boldsymbol{\omega} = \omega_2 \boldsymbol{i}' + \boldsymbol{\omega}_1$$

Therefore,

$$\begin{aligned} \dot{\boldsymbol{\omega}} &= \dot{\omega}_2 \boldsymbol{i}' + \omega_2 \dot{\boldsymbol{i}}' + \dot{\boldsymbol{\omega}}_1 \\ &= 5\boldsymbol{i}' + 5(\boldsymbol{\omega}_1 \times \boldsymbol{i}') - 10\boldsymbol{k} \\ &= 5\boldsymbol{i}' + 5(-10\boldsymbol{k}) \times \boldsymbol{i}' - 10\boldsymbol{k} \end{aligned}$$

At the instant of interest, $\boldsymbol{i}' = \boldsymbol{i}$. Hence,

$$\dot{\boldsymbol{\omega}} = 5\boldsymbol{i} - 50\boldsymbol{j} - 10\boldsymbol{k} \text{ rad/sec}^2$$

Hence, we use the above $\dot{\boldsymbol{\omega}}$ in part B of Example 15.12 to compute $\boldsymbol{V}_{XYZ}$ and $\boldsymbol{a}_{XYZ}$. The computation of $\ddot{\boldsymbol{R}}$ is straightforward and so we can compute $\boldsymbol{a}_{XYZ}$ accordingly. We leave the details to the reader.

Example 15.14[11]

To simulate the flight conditions of a space vehicle, engineers have developed the *centrifuge*, shown diagrammatically in Fig. 15.33. A main *arm*, 40 ft long, rotates about the *A–A* axis. The pilot sits in a *cockpit*, which can rotate about axis *C–C*. The *seat* for the pilot can rotate inside the cockpit about an axis shown as *B–B*. These rotations are controlled by a computer that is set to simulate certain maneuvers corresponding to the entry and exit from the earth's atmosphere, malfunctions of the control system, and so on. When a pilot sits in the cockpit, his/her head has a position which is 3 ft from the seat as shown in Fig. 15.33. At the instant of interest the main arm is rotating at 10 rpm and accelerating at 5 rpm^2. The cockpit is rotating at a constant speed about *C–C* relative to the main arm at 10 rpm. Finally, the seat is rotating at a constant speed of 5 rpm relative to the cockpit about axis *B–B*. How many *g*'s of acceleration relative to the ground is the pilot's head subject to?[12] Note that the three axes, *A–A*, *C–C*, and *B–B*, are orthogonal to each other at time *t*.

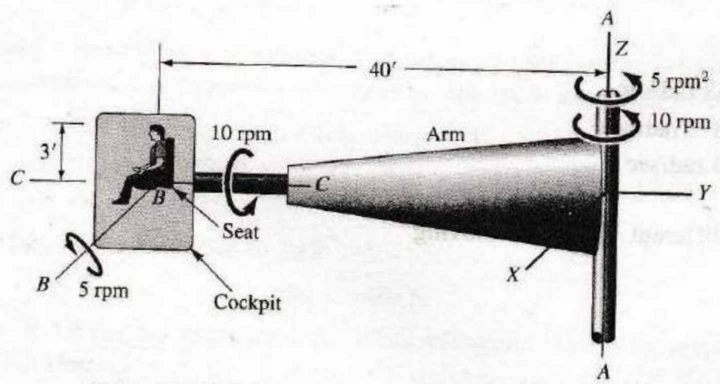


Figure 15.33. Centrifuge for simulating flight conditions.

[11]Example 15.14

Example 15.14 (Continued)

In Fig. 15.34 the arm of the centrifuge rotates relative to the ground at an angular velocity of ω_1. The cockpit meanwhile rotates relative to the arm with angular speed ω_2. Finally, the seat rotates relative to the cockpit at an angular speed ω_3. For constant ω_2, we see that, because of bearings in the arm, the vector $\boldsymbol{\omega}_2$ is "fixed" in the arm. Also, for constant ω_3, because of bearings in the cockpit, the vector $\boldsymbol{\omega}_3$ is "fixed" in the cockpit. Before we examine the acceleration of the pilot's head, note that at the instant of interest:

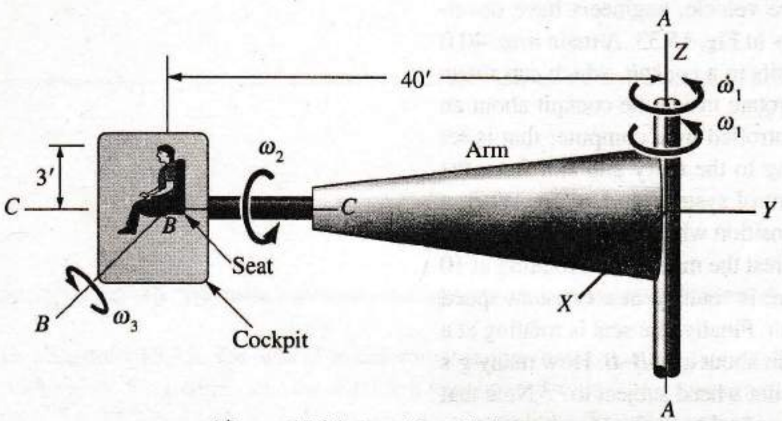


Figure 15.34. Centrifuge listing **ω**'s.

$$\omega_1 = \omega_2 = 10 \text{ rpm} = 1.048 \text{ rad/sec}$$
$$\dot{\omega}_1 = 5 \text{ rpm}^2 = .00873 \text{ rad/sec}^2$$
$$\omega_3 = 5 \text{ rpm} = .524 \text{ rad/sec}$$

We shall do this problem using three different kinds of moving references *xyz*.

ANALYSIS I

Fix *xyz* to arm.
Fix *XYZ* to ground.

Note in Fig. 15.35 that *xyz* and the arm to which it is fixed are shown dark. Note also that the axes *xyz* and *XYZ* are parallel to each other at the instant of interest.

Example 15.14 (Continued)

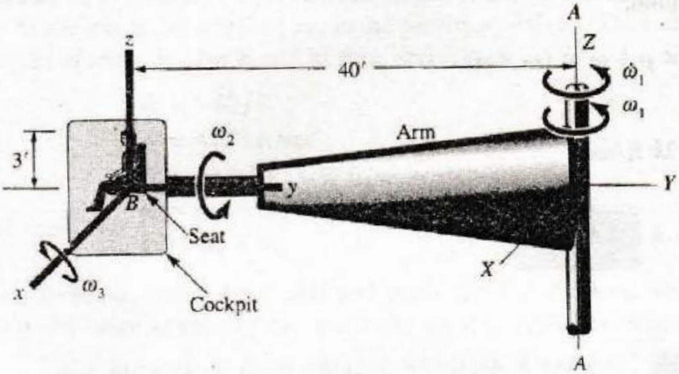


Figure 15.35. Centrifuge with xyz fixed to arm.

A. Motion of particle relative to $\boldsymbol{xyz}$

$$\boldsymbol{\rho} = 3\boldsymbol{k} \text{ ft}$$

Note that $\boldsymbol{\rho}$ is "fixed" to the seat and that the seat has an angular velocity of $(\boldsymbol{\omega}_2 + \boldsymbol{\omega}_3)$ relative to the arm and thus to xyz. Hence,

$$\begin{aligned} \boldsymbol{V}_{xyz} &= (\boldsymbol{\omega}_2 + \boldsymbol{\omega}_3) \times \boldsymbol{\rho} \\ &= (1.048\boldsymbol{j} + .524\boldsymbol{i}) \times 3\boldsymbol{k} = 3.14\boldsymbol{i} - 1.572\boldsymbol{j} \text{ ft/sec} \end{aligned}$$

$$\boldsymbol{a}_{xyz} = \left(\frac{d\boldsymbol{V}_{xyz}}{dt}\right)_{xyz} = \left[\frac{d}{dt_{xyz}}(\boldsymbol{\omega}_2 + \boldsymbol{\omega}_3)\right] \times \boldsymbol{\rho} + (\boldsymbol{\omega}_2 + \boldsymbol{\omega}_3) \times \left(\frac{d\boldsymbol{\rho}}{dt}\right)_{xyz}$$

Clearly, relative to the arm, and thus to xyz, $\boldsymbol{\omega}_2$ is constant. And $\boldsymbol{\omega}_3$ is fixed in the cockpit that has an angular velocity of $\boldsymbol{\omega}_2$ relative to xyz. Thus, we have

$$\begin{aligned} \boldsymbol{a}_{xyz} &= (\boldsymbol{0} + \boldsymbol{\omega}_2 \times \boldsymbol{\omega}_3) \times \boldsymbol{\rho} + (\boldsymbol{\omega}_2 + \boldsymbol{\omega}_3) \times \boldsymbol{V}_{xyz} \\ &= (1.048\boldsymbol{j} \times .524\boldsymbol{i}) \times 3\boldsymbol{k} + (1.048\boldsymbol{j} + .524\boldsymbol{i}) \times (3.14\boldsymbol{i} - 1.572\boldsymbol{j}) \\ &= -4.12\boldsymbol{k} \text{ ft/sec}^2 \end{aligned}$$

B. Motion of $\boldsymbol{xyz}$ **relative to** $\boldsymbol{XYZ}$

$$\boldsymbol{R} = -40\boldsymbol{j} \text{ ft}$$

Note that $\boldsymbol{R}$ is fixed in the arm, which has an angular velocity $\boldsymbol{\omega}_1$ relative to XYZ. Hence,

$$\begin{aligned} \dot{\boldsymbol{R}} &= \boldsymbol{\omega}_1 \times \boldsymbol{R} = 1.048\boldsymbol{k} \times (-40\boldsymbol{j}) = 41.9\boldsymbol{i} \text{ ft/sec} \\ \ddot{\boldsymbol{R}} &= \boldsymbol{\omega}_1 \times \dot{\boldsymbol{R}} + \dot{\boldsymbol{\omega}}_1 \times \boldsymbol{R} \\ &= 1.048\boldsymbol{k} \times 41.9\boldsymbol{i} + .00873\boldsymbol{k} \times (-40\boldsymbol{j}) \\ &= 43.9\boldsymbol{j} + .349\boldsymbol{i} \text{ ft/sec}^2 \\ \boldsymbol{\omega} &= \boldsymbol{\omega}_1 = 1.048\boldsymbol{k} \text{ rad/sec} \\ \dot{\boldsymbol{\omega}} &= \dot{\boldsymbol{\omega}}_1 = .00873\boldsymbol{k} \text{ rad/sec}^2 \end{aligned}$$

Example 15.14 (Continued)

We can now substitute into the following equation:

$$\boldsymbol{a}_{XYZ} = \boldsymbol{a}_{xyz} + \ddot{\boldsymbol{R}} + 2\boldsymbol{\omega} \times \boldsymbol{V}_{xyz} + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho})$$

Therefore,

$$\boldsymbol{a}_{XYZ} = 3.64\boldsymbol{i} + 50.5\boldsymbol{j} - 4.12\boldsymbol{k} \text{ ft/sec}^2$$

$$|\boldsymbol{a}_{XYZ}| = \frac{\sqrt{3.64^2 + 50.5^2 + 4.12^2}}{32.2} = \boxed{1.578\,g}$$

ANALYSIS II

Fix *xyz* to cockpit.
Fix *XYZ* to ground.

This situation is shown in Fig. 15.36.

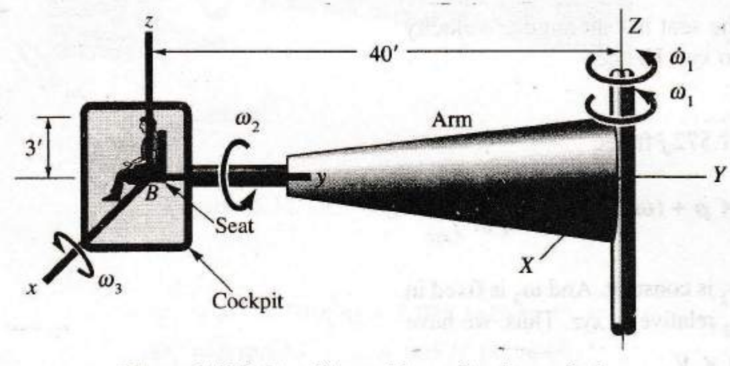


Figure 15.36. Centrifuge with *xyz* fixed to cockpit.

A. Motion of particle relative to *xyz*

$$\boldsymbol{\rho} = 3\boldsymbol{k} \text{ ft}$$

Note that $\boldsymbol{\rho}$ is fixed to the seat, which has an angular velocity of $\boldsymbol{\omega}_3$ relative to the cockpit and thus relative to *xyz*. Hence,

$$\boldsymbol{V}_{xyz} = \boldsymbol{\omega}_3 \times \boldsymbol{\rho} = .524\boldsymbol{i} \times 3\boldsymbol{k} = -1.572\boldsymbol{j} \text{ ft/sec}$$

$$\boldsymbol{a}_{xyz} = \left(\frac{d\boldsymbol{\omega}_3}{dt}\right)_{xyz} \times \boldsymbol{\rho} + \boldsymbol{\omega}_3 \times \left(\frac{d\boldsymbol{\rho}}{dt}\right)_{xyz}$$

But $\boldsymbol{\omega}_3$ is constant as seen from the cockpit and thus from *xyz*. Hence,

$$\boldsymbol{a}_{xyz} = \boldsymbol{0} \times \boldsymbol{\rho} + \boldsymbol{\omega}_3 \times \boldsymbol{V}_{xyz} = .524\boldsymbol{i} \times (-1.572\boldsymbol{j})$$
$$= -.824\boldsymbol{k} \text{ ft/sec}^2$$

Example 15.14 (Continued)

B. Motion of *xyz* relative to *XYZ*. The origin of *xyz* in this analysis has the same motion as the origin of *xyz* in the previous analysis. Thus, we use the results of analysis I for $\boldsymbol{R}$ and its time derivatives.

$$\boldsymbol{R} = -40\boldsymbol{j} \text{ ft}$$
$$\dot{\boldsymbol{R}} = 41.9\boldsymbol{i} \text{ ft/sec}$$
$$\ddot{\boldsymbol{R}} = 43.9\boldsymbol{j} + .349\boldsymbol{i} \text{ ft/sec}^2$$
$$\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 = 1.048\boldsymbol{j} + 1.048\boldsymbol{k} \text{ rad/sec}$$
$$\dot{\boldsymbol{\omega}} = \dot{\boldsymbol{\omega}}_1 + \dot{\boldsymbol{\omega}}_2$$

We are given $\dot{\boldsymbol{\omega}}_1$ about the Z axis and $\boldsymbol{\omega}_2$ is fixed in the arm, which is rotating with angular velocity $\boldsymbol{\omega}_1$ relative to the *XYZ* reference. Hence,

$$\dot{\boldsymbol{\omega}} = \dot{\boldsymbol{\omega}}_1 + \boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2 = .00873\boldsymbol{k} + (1.048\boldsymbol{k} \times 1.048\boldsymbol{j})$$
$$= -1.098\boldsymbol{i} + .00873\boldsymbol{k} \text{ rad/sec}^2$$

We can now substitute into the key equation, 15.24:

$$\boldsymbol{a}_{XYZ} = \boldsymbol{a}_{xyz} + \ddot{\boldsymbol{R}} + 2\boldsymbol{\omega} \times \boldsymbol{V}_{xyz} + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho})$$
$$= 3.64\boldsymbol{i} + 50.5\boldsymbol{j} - 4.12\boldsymbol{k} \text{ ft/sec}^2$$

$$|\boldsymbol{a}_{XYZ}| = 1.578g$$

ANALYSIS III

Fix *xyz* to seat.
Fix *XYZ* to ground.

This situation is shown in Fig. 15.37.

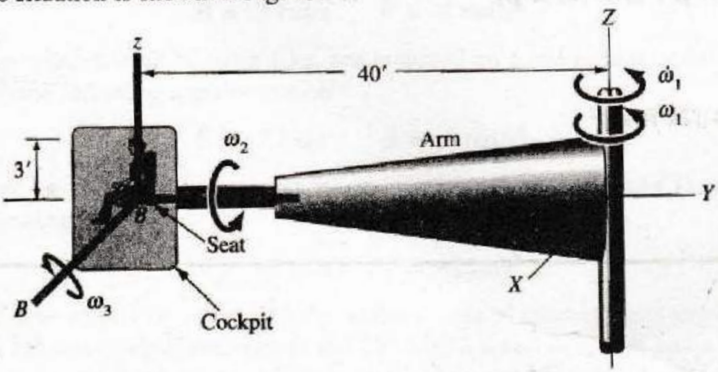


Figure 15.37. Centrifuge with *xyz* fixed to seat.

Example 15.14 (Continued)

A. Motion of particle relative to *xyz*

$$\boldsymbol{\rho} = 3\boldsymbol{k}\ \text{ft}$$

Since the particle is fixed to the seat and is thus fixed in *xyz*, we can say:

$$\boldsymbol{V}_{xyz} = \boldsymbol{0}$$
$$\boldsymbol{a}_{xyz} = \boldsymbol{0}$$

B. Motion of *xyz* relative to *XYZ*. Again, the origin of *xyz* has identically the same motion as in the previous analyses. Thus, we have the same results as before for $\boldsymbol{R}$ and its derivatives.

$$\boldsymbol{R} = -40\boldsymbol{j}\ \text{ft}$$
$$\dot{\boldsymbol{R}} = 41.9\boldsymbol{i}\ \text{ft/sec}$$
$$\ddot{\boldsymbol{R}} = 43.9\boldsymbol{j} + .349\boldsymbol{i}\ \text{ft/sec}^2$$
$$\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 + \boldsymbol{\omega}_3 = 1.048\boldsymbol{k} + 1.048\boldsymbol{j} + .524\boldsymbol{i}$$
$$\dot{\boldsymbol{\omega}} = \dot{\boldsymbol{\omega}}_1 + \dot{\boldsymbol{\omega}}_2 + \dot{\boldsymbol{\omega}}_3$$

Note that $\dot{\boldsymbol{\omega}}_1$ is given. Also, $\boldsymbol{\omega}_2$ is fixed in the arm, which rotates with angular speed $\boldsymbol{\omega}_1$ relative to *XYZ*. Finally, $\boldsymbol{\omega}_3$ is fixed in the cockpit, which has an angular velocity $\boldsymbol{\omega}_2 + \boldsymbol{\omega}_1$ relative to *XYZ*. Thus,

$$\begin{aligned}\dot{\boldsymbol{\omega}} &= \dot{\boldsymbol{\omega}}_2 + \boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2 + (\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2) \times \boldsymbol{\omega}_3 \\ &= .00873\boldsymbol{k} + (1.048\boldsymbol{k} \times 1.048\boldsymbol{j}) + (1.048\boldsymbol{k} + 1.048\boldsymbol{j}) \times (.524\boldsymbol{i}) \\ &= -1.098\boldsymbol{i} + .549\boldsymbol{j} - .540\boldsymbol{k}\end{aligned}$$

We now go to the basic equation, 15.24.

$$\boldsymbol{a}_{XYZ} = \boldsymbol{a}_{xyz} + \ddot{\boldsymbol{R}} + 2\boldsymbol{\omega} \times \boldsymbol{V}_{xyz} + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho})$$

Substituting, we get

$$\boldsymbol{a}_{XYZ} = 3.64\boldsymbol{i} + 50.5\boldsymbol{j} - 4.12\boldsymbol{k}\ \text{ft/sec}^2$$

$$|\boldsymbol{a}_{XYZ}| = \mathbf{1.578}\boldsymbol{g}$$

Example 15.15

A submersible (see Fig. 15.38) is moving relative to the ground reference XYZ so as to have the following motion at the instant of interest for point A fixed to shaft $\overline{CD}$ which in turn is fixed to the submersible:

$$\boldsymbol{V} = 3\boldsymbol{i} + .6\boldsymbol{j} \text{ m/s}$$
$$\boldsymbol{a} = 2\boldsymbol{i} + 3\boldsymbol{j} - .5\boldsymbol{k} \text{ m/s}^2$$

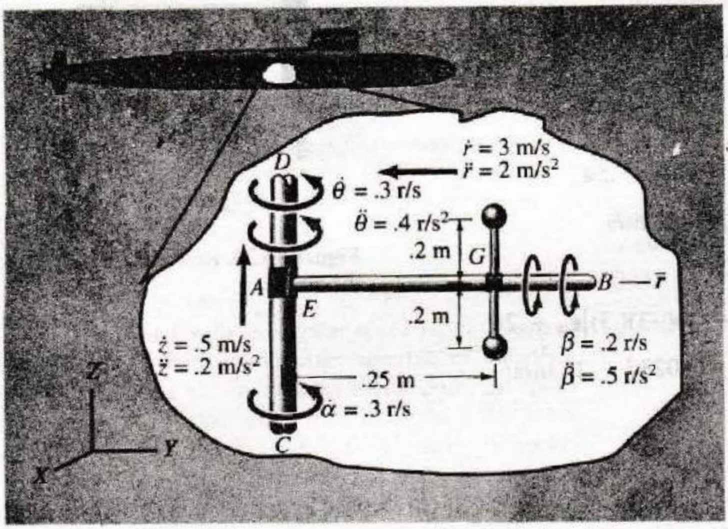


Figure 15.38. Rotating device inside a moving submersible. $\overline{CD}$ is fixed to the submersible.

At the instant of interest, the vessel has an angular speed of rotation $\dot{\alpha}$ = .3 rad/s about the centerline of $\overline{CD}$ as seen from the ground reference. A horizontal rod $\overline{EB}$ has the following angular motion about $\overline{CD}$:

$$\dot{\theta} = .3 \text{ rad/s} \qquad \ddot{\theta} = .4 \text{ rad/s}^2$$

Two spheres, each of mass 1 kg, are mounted on a rod turning about $\overline{EB}$ with the following angular motion

$$\dot{\beta} = .2 \text{ rad/s} \qquad \ddot{\beta} = .5 \text{ rad/s}^2$$

Also, the rod and the attached spherical masses advance toward $\overline{CD}$ at the following rate:

$$\dot{r} = -3 \text{ m/s} \qquad \ddot{r} = -2 \text{ m/s}^2$$

at a time when r = .25 m. Finally, at the instant of interest, the horizontal rod $\overline{EB}$ moves up along vertical rod $\overline{CD}$ with a speed of .5 m/s and a rate of change of speed of .2 m/s². What force must rod $\overline{EB}$ exert at point G at this instant *due only to the motion of the two spheres*?

Example 15.15 (Continued)

We will first consider the motion of the *center of mass* of the rotating spheres which clearly must be G. We now proceed to get the acceleration of G relative to XYZ (see Fig. 15.39).

Fix xyz to the vessel at A.
Fix XYZ to the ground.

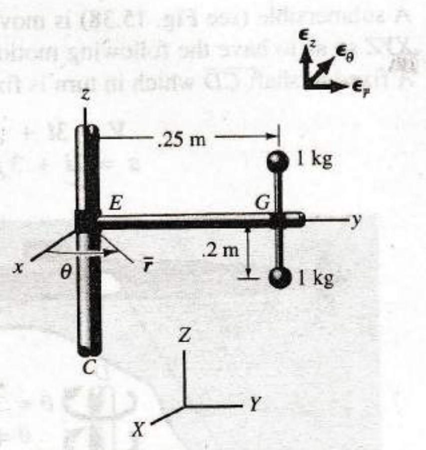


Figure 15.39. Reference xyz fixed to the vessel.

A. Motion of G relative to xyz (using cylindrical coordinates). Use Figs. 15.38 and 15.39.

$$\boldsymbol{\rho} = .25\boldsymbol{\epsilon}_{\bar{r}} = .25\boldsymbol{j}\text{ m}$$

$$\boldsymbol{V}_{xyz} = \dot{\bar{r}}\boldsymbol{\epsilon}_{\bar{r}} + \bar{r}\dot{\theta}\boldsymbol{\epsilon}_\theta + \dot{z}\boldsymbol{\epsilon}_z = -3\boldsymbol{\epsilon}_{\bar{r}} + (.25)(.3)\boldsymbol{\epsilon}_\theta + .5\boldsymbol{\epsilon}_z$$
$$= -3\boldsymbol{j} - .075\boldsymbol{i} + .5\boldsymbol{k} = -.075\boldsymbol{i} - 3\boldsymbol{j} + .5\boldsymbol{k}\text{ m/s}$$

$$\boldsymbol{a}_{xyz} = (\ddot{\bar{r}} - \bar{r}\dot{\theta}^2)\boldsymbol{\epsilon}_{\bar{r}} + (\bar{r}\ddot{\theta} + 2\dot{\bar{r}}\dot{\theta})\boldsymbol{\epsilon}_\theta + \ddot{z}\boldsymbol{\epsilon}_z$$
$$= \left[-2 - (.25)(.3)^2\right]\boldsymbol{\epsilon}_{\bar{r}} + \left[(.25)(.4) + (2)(-3)(.3)\right]\boldsymbol{\epsilon}_\theta + .2\boldsymbol{\epsilon}$$
$$= -2.023\boldsymbol{\epsilon}_{\bar{r}} - 1.7\boldsymbol{\epsilon}_\theta + .2\boldsymbol{\epsilon}_z = 1.7\boldsymbol{i} - 2.023\boldsymbol{j} + .2\boldsymbol{k}\text{ m/s}^2$$

B. Motion of xyz relative to XYZ

$$\dot{\boldsymbol{R}} = 3\boldsymbol{i} + .6\boldsymbol{j}\text{ m/s}$$
$$\ddot{\boldsymbol{R}} = 2\boldsymbol{i} + 3\boldsymbol{j} - .5\boldsymbol{k}\text{ m/s}^2$$
$$\boldsymbol{\omega} = .3\boldsymbol{k}\text{ rad/s}$$
$$\dot{\boldsymbol{\omega}} = \boldsymbol{0}\text{ rad/s}^2$$

We may now express $\boldsymbol{a}_{XYZ}$ for point G. Thus

$$\boldsymbol{a}_{XYZ} = \boldsymbol{a}_{xyz} + \ddot{\boldsymbol{R}} + 2\boldsymbol{\omega} \times \boldsymbol{V}_{xyz} + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho})$$
$$= (1.7\boldsymbol{i} - 2.023\boldsymbol{j} + .2\boldsymbol{k}) + (2\boldsymbol{i} + 3\boldsymbol{j} - .5\boldsymbol{k}) + 2(.3\boldsymbol{k})$$
$$\times (-.075\boldsymbol{i} - 3\boldsymbol{j} + .5\boldsymbol{k}) + \boldsymbol{0} \times \boldsymbol{\rho} + (.3\boldsymbol{k}) \times (.3\boldsymbol{k} \times .25\boldsymbol{j})$$
$$= 5.5\boldsymbol{i} + .9095\boldsymbol{j} - .3\boldsymbol{k}\text{ m/s}^2$$

Now we apply **Newton's law** to the mass center G at the instant of interest. Denoting the force from the rod AB onto G as $\boldsymbol{F}_{\text{ROD}}$, we get

$$\boldsymbol{F}_{\text{ROD}} - 2\text{mg}\boldsymbol{k} = 5.5\boldsymbol{i} + .9095\boldsymbol{j} - .3\boldsymbol{k}$$
$$\therefore \boldsymbol{F}_{\text{ROD}} = (2)(1)(9.81)\boldsymbol{k} + 5.5\boldsymbol{i} + .9095\boldsymbol{j} - .3\boldsymbol{k}$$

$$\boldsymbol{F}_{\text{ROD}} = 5.5\boldsymbol{i} + .9095\boldsymbol{j} + 19.32\boldsymbol{k}\text{ N}$$

This is our desired result.