Chapter 17

Energy & Impulse - Momentum Methods for Rigid bodies.

Part A Energy Method.

Kinetic Energy of a Rigid body

Kinetic Energy of an aggregate of particles relative to any reference is the sum of two parts, which we list again as:

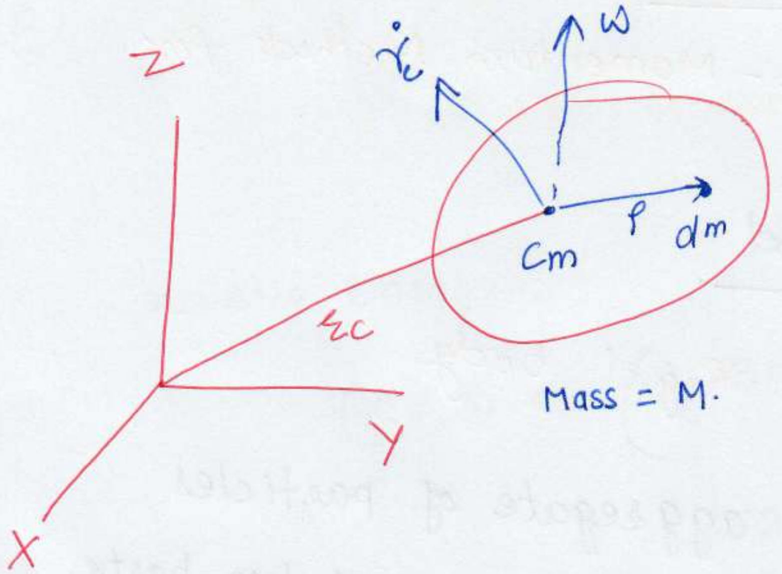
1. Kinetic Energy of a hypothetical particle that has a mass equal to the total mass of the system and a motion corresponding to that of the mass center of the system, plus
2. The kinetic energy of the particles relative to the mass center.

$$K.E. \quad \frac{1}{2} M |\dot{r}_c|^2 + \frac{1}{2} \sum_{i=1}^{n} m_i |\dot{\rho}_i|^2$$

Where ρ_i is the position vector from the mass center to the ith particle.

Rigid body $\Rightarrow$ aggregate of particles.

For this case: Velocity of any particle relative to the mass center $\quad \dot{\rho}_i = \omega \times \rho_i$



where ω = angular velocity of the body relative to the reference XYZ in which we are computing the kinetic energy.

For a Rigid body

$$|\dot{r}_c|^2 = V_c^2$$

$$K.E = \frac{1}{2} M V_c^2 + \frac{1}{2} \iiint_m |\omega \times \rho|^2 dm$$

Where ρ represents the position vector from the center of mass to any element of mass dm.

$$\iiint_M |\omega \times \rho|^2 dm = \iiint (\omega \times \rho) \cdot (\omega \times \rho)\, dm$$

$$\Rightarrow \iiint [(\omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}) \times (x\hat{i} + y\hat{j} + z\hat{k})] \cdot [(\omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}) \times (x\hat{i} + y\hat{j} + z\hat{k})]\, dm$$

$$\Rightarrow \left[\iiint_m \overbrace{(z^2+y^2)\,dm}^{\Rightarrow I_{xx}}\right]\omega_x^2 - \left(\iiint_M \overbrace{xy\,dm}^{\Rightarrow I_{xy}}\right)\omega_x\omega_y - \left(\iiint_M \overbrace{xz\,dm}^{I_{xz}}\right)\omega_x\omega_z$$

$$-\left[\iiint_M \overbrace{xy\,dm}^{I_{xy}}\right]\omega_x\omega_y + \left[\iiint_m \overbrace{(x^2+z^2)\,dm}^{\Rightarrow I_{yy}}\right]\omega_y^2 - \left[\iiint_M \overbrace{yz\,dm}^{\Rightarrow I_{yz}}\right]\omega_y\omega_z$$

$$-\left[\iiint_m \overbrace{zx\,dm}^{I_{zx}}\right]\omega_z\omega_x - \left[\iiint_m \overbrace{zy\,dm}^{I_{zy}}\right]\omega_z\omega_y + \left[\iiint_M \overbrace{(x^2+y^2)\,dm}^{I_{zz}}\right]\omega_z^2$$

$$\iiint_M |\omega\times\rho|^2\,dm = I_{xx}\omega_x^2 - I_{xy}\omega_{xy} - I_{xz}\omega_x\omega_z$$
$$- I_{yx}\omega_x\omega_y + I_{yy}\omega_y^2 - I_{yz}\omega_y\omega_z$$
$$\underbrace{- I_{zx}\omega_z\omega_x - I_{zy}\omega_z\omega_y + I_{zz}\omega_z^2}$$

Now

$$K.E = \frac{1}{2}MV_c^2 + \frac{1}{2}$$

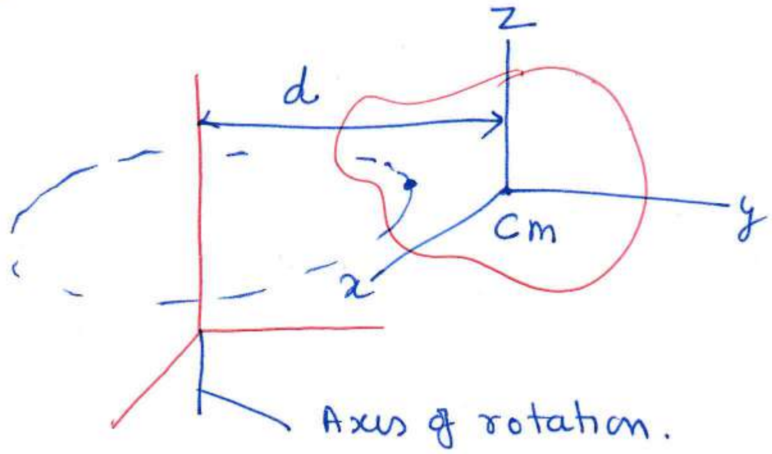
If principal axes are chosen

$$K.E = \frac{1}{2}MV_c^2 + \underbrace{\frac{1}{2}\left(I_{xx}\omega_x^2 + I_{yy}\omega_y^2 + I_{zz}\omega_z^2\right)}$$

Note:

Have same form as the kinetic energy term that is due to translation, with the moment of inertia

Rigid body Undergoing pure rotation about Z- axis



→ Calculate kinetic Energy:-

Z- axis is collinear with the ω vector and the axis of rotation.. The reference xyz at center of mass is chosen parallel to XYZ.

$$\omega_x = \omega_y = 0, \quad \omega_z = \omega.$$

$$K.E = \frac{1}{2} M v_c^2 + \frac{1}{2} I_{zz} \omega^2$$

~~$V_c = \omega \times d$, Where d is where~~

$\underline{V_c = \omega d}$, where d is the distance between the axis of rotation z and z axis at the center of mass

$$K.E = \frac{1}{2} m (\omega^2 d^2) + \frac{1}{2} I_{zz} \omega^2$$

$$= \frac{1}{2} (\underbrace{I_{zz} + m d^2}) \omega^2 \Rightarrow \frac{1}{2} I \omega^2$$

Moment of Inertia of ~~abo~~ the body about the axis of rotation Z.

For a body undergoing general plane motion parallel to the XY plane, where xyz are taken at center of mass and oriented parallel to XYZ, we get

$$K.E. = \frac{1}{2} m v_c^2 + \frac{1}{2} I_{zz} \omega_z^2$$

Example 17.1

Compute the kinetic energy of the crank as shown in fig.

Piston A = 1 kg C

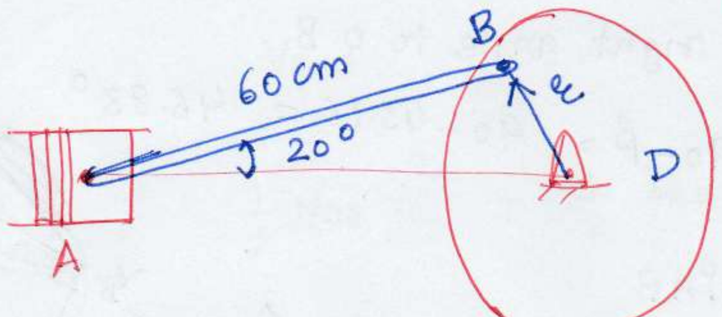
Rod AB = 60 cm, 2 kg

Flywheel D = 50 kg

Radius of gyration = 45 cm

The radius r = 30 cm

At the instant of intrest, piston A is moving to the right at a speed of V of 300 cm/sec.



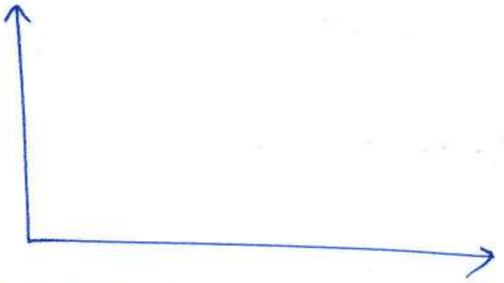
Piston A = Translatory

Rod AB = Plane motion

Flywheel D = Pure Rotation.

(A) Kinetic Energy of Piston

$(KE)_A = \frac{1}{2} M V^2 = \frac{1}{2} 1 \times (3)^2 \Rightarrow \frac{9}{2} = 4.5 \ kg \ m^2/s^2 \ (J)$

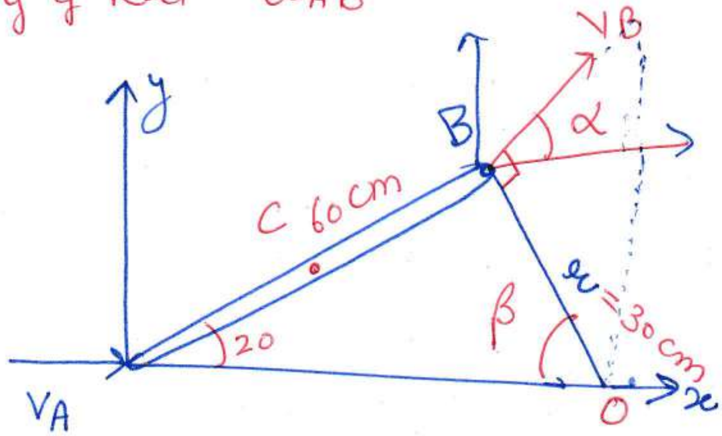


(B) Kinetic Energy for Rod AB

⇒ mass center of Rod

= Velocity of mass center – translation, V_c

= angular velocity of Rod ω_{AB}



$\frac{60}{\sin\beta} = \frac{30}{\sin 20}$

$\sin\beta = \frac{\cancel{60}^{\,2}}{\cancel{30}} \sin 20$

$= .68404$

$\beta = \sin^{-1}(.68404) = 43.16^\circ$

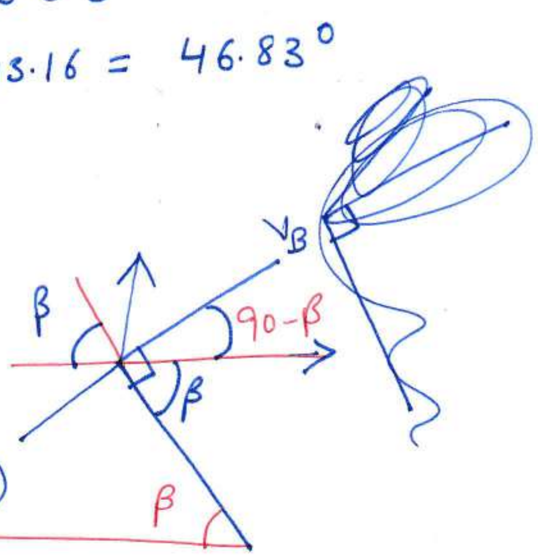
Velocity of B is at right angle to O B

$\alpha = 90 - \beta = 90 - 43.16 = 46.83^\circ$

$V_B = V_A + (\omega_{AB} K) \times \rho_{AB}$

$V_B(\cos\alpha \hat{i} + \sin\alpha \hat{j}) = 3\hat{i} + (\omega_{AB} k \times$

$(.6 (\cos 20 \hat{i} + \sin 20 \hat{j}))$



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$$V_B\cos\alpha\,\hat{i} + V_B\sin\alpha\,\hat{j} = 3\hat{i} + .6\,\omega_{AB}\cos 20\,\hat{j} - .6\,\omega_{AB}\sin 20\,\hat{i}$$

$$V_B \times .6841 = 3 - 0.2051\,\omega_{AB} \Rightarrow .6841\,V_B + 0.2051\,\omega_{AB} = 3 \quad \text{— (A)}$$

$$.7293\,V_B = .5638\,\omega_{AB}$$

$$v_B = .773\,\omega_{AB} \quad \text{— (B)}$$

$$.6841 \times .773\,\omega_{AB} + .2051\,\omega_{AB} = 3$$

$$.7339\,\omega_{AB} = 3 \Rightarrow \boxed{\begin{aligned}\omega_{AB} &= 4.087 \text{ rad/sec}\\ v_B &= 3.15 \text{ m/sec.}\end{aligned}}$$

(C) Velocity of mass center $\underline{C}$ of AB.

$$V_C = V_A + (\omega_{AB}\hat{k}) \times \rho_{AC}$$

~~V_C (cos 20~~

$$V_C = 10\hat{i} + 4.087\,\hat{k} \times (.3 \times (\cos 20\,\hat{i} + \sin 20\,\hat{j}))$$

$$V_C = 10\hat{i} + 1.152\,\hat{j} - .419\,\hat{i}$$

$$V_C = 9.58\,\hat{i} + 1.152\,\hat{j}$$

$$\boxed{(K.E)_{AB} = \frac{1}{2} M_{AB} V_C^2 + \frac{1}{2} I_{zz}\,\omega_{AB}^2}$$

$$= \frac{1}{2} \times 2 \times (9.58^2 + 1.152^2) + \frac{1}{2}\left(\frac{2 \times .6^2}{12}\right) 4.087^2$$

$$(K.E)_{AB} = 93.10 + \cancel{0.05}\ 0.5011 \Rightarrow 93.601 \,(\text{kg m}^2/\text{sec}^2)$$

(J) Joule

D Find out the angular speed

$$V_B = \omega_D r \Rightarrow \omega_D = \frac{V_B}{r} = \frac{3.15}{.3} = 10.5 \text{ rad/sec.}$$

Kinetic Energy

$$(KE)_D = \left(\frac{1}{2} 50 \times (.45)^2\right) \times (10.5)^2$$

$$= 558.14 \text{ (J)}$$

Total K.E

$$K.E = (KE)_A + (KE)_{AB} + (KE)_D$$

$$= 4.5 + 93.61 + 558.14$$

$$= 656.25 \text{ (J)}$$

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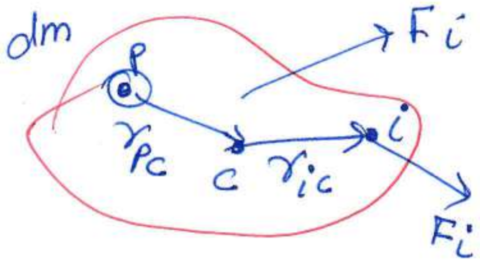
Work-Energy Relation for a Rigid body

The rate of work $\dot{W}$ of the external forces and the kinetic energy T of a rigid body can be expressed as 3

$$\dot{W} = \sum_i \bar{F}_i \cdot \bar{v}_i = \sum_i F_i \cdot (\bar{v}_c + \bar{\omega} \times \bar{r}_{ic})$$

$$= \underbrace{\left(\sum_i \bar{F}_i\right) \cdot \bar{v}_c + \bar{\omega} \cdot \left(\sum_i \bar{r}_{ic} \times \bar{F}_i\right)}$$

$$\dot{W} = \boxed{\bar{F} \cdot \vec{V}_c + \bar{M}_c \cdot \omega}$$



$$T = \frac{1}{2} m v_c^2 + \frac{1}{2} \int_m v_{pc}^2 \, dm$$

$$= \frac{1}{2} m \bar{v}_c \cdot \bar{v}_c + \frac{1}{2} \int_m \bar{v}_{pc} \cdot \bar{v}_{pc} \, dm$$

$$\dot{T} = \frac{1}{2} m \bar{v}_c \cdot \dot{\bar{v}}_c + \frac{1}{2} m \dot{\bar{v}}_c \cdot \bar{v}_c + \frac{1}{2} \left[\int_m \bar{v}_{pc} \cdot \dot{\bar{v}}_{pc} \, dm + \int \dot{\bar{v}}_{pc} \cdot \bar{v}_{pc} \, dm \right]$$

$$\Rightarrow \boxed{m \bar{a}_c \cdot \bar{v}_c + \int_m \bar{a}_{pc} \cdot \bar{v}_{pc} \, dm}$$

$$\Rightarrow F \cdot \bar{v}_c + \int_m a_{pc} \cdot (\bar{\omega} \times \vec{r}_{pc}) \, dm$$

$$= \bar{F} \cdot \bar{v}_c + \bar{\omega} \cdot \int_m (r_{pc} \times a_{pc}) \, dm$$

$$= \bar{F} \cdot \bar{v}_c + \bar{\omega} \cdot \underbrace{\frac{d}{dt} \left[\int_m (r_{pc} \times v_{pc}) \, dm \right]}_{\dot{H}_c} \Rightarrow \bar{F} \cdot \bar{v}_c + \bar{\omega} \cdot \dot{H}_c$$

$$\Rightarrow \boxed{\bar{F} \cdot \bar{v}_c + \omega \cdot \bar{M}_c}$$

Hence we obtain the following work energy relations in rate form and in integrated form

$$\mathring{T} = \mathring{W}, \quad T_2 - T_1 = W_{1-2}$$

Where

T_1, T_2 are the values of kinetic energy in configurations 1 & 2.

W_c = Work done by the conservative forces

W_{nc} = Work done by the non-conservative forces.

$$\mathring{W} = \mathring{W}_c + \mathring{W}_{nc} = -\dot{V} + \mathring{W}_{nc}$$

$$\boxed{\dot{T} + \dot{V} = \dot{W}_{nc}}, \quad \boxed{(T_2 + V_2) - (T_1 + V_1) = W_{nc\,1-2}}$$

Where V_1, V_2 - are the values of ~~kine~~ potential energy in configuration 1 & 2. if all the forces are conservative the $\mathring{W}_{nc} = 0$

$$\boxed{\mathring{T} + \mathring{V} = 0, \quad T_2 + V_2 = T_1 + V_1}$$

Conservative Forces

A force $\bar{F}$ is called conservative if the work done by it from t_1 to t_2, during the motion of its material point of application from location $\vec{r}_1$ to $\vec{r}_2$, is independent of the path C, connecting these locations.

$$W_{1-2} = \int_{t_1}^{t_2} \bar{F} \cdot \bar{v}\, dt = \int_{C\,\bar{r}_1}^{\bar{r}_2} \bar{F} \cdot d\bar{r} \quad \forall\, C,\ \& \quad \forall\, \bar{r}_1, \bar{r}_2 \qquad \text{(A)}$$

Hence conservative forces acts on the same material point of the body and is independent of ~~its~~ its velocity and time. $\bar{F} = \underline{F}(r)$

Equation (A) implies that the integrand is a perfect differential of a position dependent scalar function $V(\underline{r})$, called the potential Energy

$$dW = \bar{F} \cdot d\bar{r} = -dV$$

$$\Rightarrow \quad W_{1-2} = -\int_{V(r_1)}^{V(r_2)} dV = -\left[V(r_2) - V(r_1)\right]$$

i.e Work done from $\vec{r}_1$ to $\bar{r}_2$ equals negative of the change in potential energy from $\bar{r}_1$ to $\bar{r}_2$

Hence work done by conservative force in any closed path is zero.

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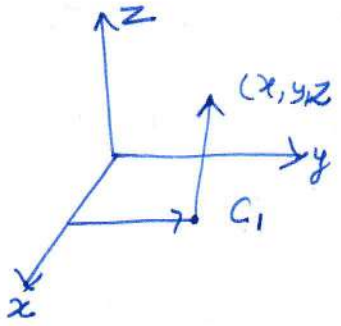
Potential Energies of some conservative forces

1. Picewise linear curve C_1



$$V(\bar{r}) - V(\bar{0}) = -\int_{C_1}^{r} \bar{F}\cdot d\bar{r}$$

$$= -\int_0^x F_x(x,0,0)\,dx$$

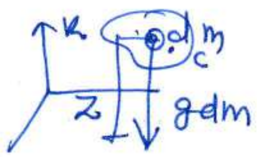
$$- \int_0^y F_y(x,y,0)\,dy - \int_0^z F_z(x,y,z)\,dz$$

2. For a constant force $\bar{F}$, with datum O

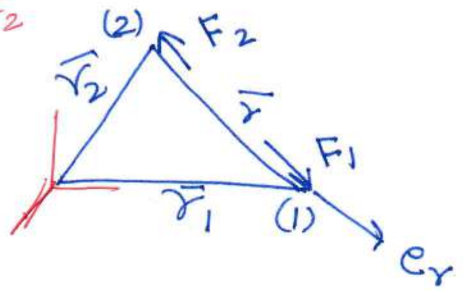
$$V(\bar{r}) = -\bar{F}\cdot\bar{r} = -F_x x - F_y y - F_z z$$

The potential energy of $-g\,dm\,\hat{k}$ is $gz\,dm$ and Total

$$V = \int_m gz\,dm = mgz_c$$



3. For a pair of mutual forces $\bar{F}_1$ and $\bar{F}_2$ with $\bar{F}_1 = -\bar{F}_2 = F(r)\,e_r$



$$dV = -(\bar{F}_1 d\bar{r}_1 + \bar{F}_2 d\bar{r}_2)$$

$$= -\bar{F}_1\cdot d(\bar{r}_1 - \bar{r}_2) = -\bar{F}_1\cdot d\bar{r}$$

$$= -\bar{F}(r)\frac{\bar{r}}{r}\cdot d\bar{r} = -F(r)\frac{d(\bar{r}\cdot\bar{r})}{2r} = -F(r)\frac{d(r^2)}{2r} = -F(r)\,dr$$

For datum at $r = r_0$, $V = \int_{r_0}^{r} F(r)\,dr$

(a) For mutual Gravitational force

$$F(r) = -\frac{GMm}{r^2} \quad \text{with datum } r_0 = \infty$$

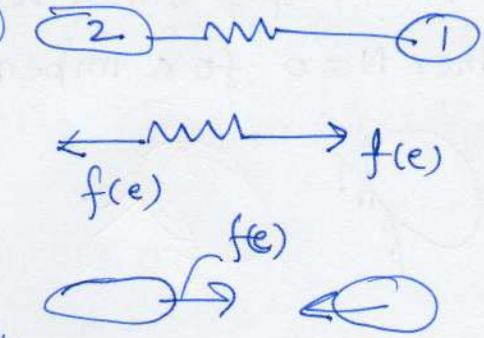
Yields $V = -\frac{GMm}{r}$ $\left(V = -\int_{r_0}^{r} F(r)\,dr \right.$

(b) For a pair of spring forces, let the pull on the spring be $f(e)$ for an extension $e = r - L_0$

Wher L_0 = unstreptched Length

Hence $F(r) = -f(e) = de = dr$ and yields,

$$V = \int_0^e f(e)\,de \quad \text{(B)}$$



With datum at unstreched configuration $r_0 = L_0$.

For a linear spring of stiffnen k

$f(e) = Ke$ and eq. (B) will yeild $V = K\frac{e^2}{2} = K\frac{\delta^2}{2}$

with $\delta (= e)$ being the extension.

(C) For torsional spring with axial couple $M_{(t)} = f(\theta)$ for relative axial twist of θ between the ends.

$V = \int_0^{\theta} f(\theta)\,d\theta$. For a linear torsional spring of torsional stiffness k_t = $M_t = k_t\theta$.

$V = k_t\frac{\theta^2}{2}$ with untwisted configuration as the datum.

Workless Forces

We list some forces which are workless and some pair of forces which together are workless though individually each may perform work.

1. Magnetic force $\bar{F} = q\bar{v}\times\bar{B}$ since $\dot{W} = \bar{F}\cdot\bar{v} = q\bar{v}\times\bar{B}\cdot\bar{v} = 0$

2. Reaction $\bar{R}_1$ at contact P_1 with a fixed body with no-slip since $\dot{W} = \bar{R}_1\cdot \bar{v}_{P_1} = \bar{R}_1\cdot\bar{0} = 0$

3. Reaction at smooth contact with a fixed body with or without slip, since $\dot{W} = N\bar{n}\cdot\bar{v} = Nv_n = 0$ as either $N=0$ for impending separation or $\bar{v}_n = 0$ for no separation

4. Reaction $\bar{R}$ at A at smooth ball and socket or smooth hinge joint with a fixed ~~f~~ body since $\dot{W} = \bar{R}\cdot\bar{v}_A = \bar{R}\cdot\bar{0} = 0$

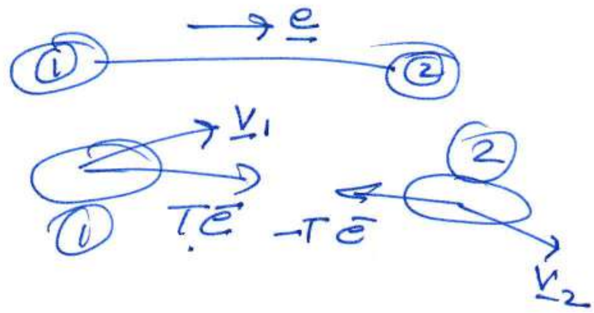
(5) Pair of reaction $\bar{R}$ and $-\bar{R}$ at A at smooth ball and socket or smooth hinge joint, since $\dot{W} = \bar{R}\cdot v_A + (-\bar{R})\bar{v}_A = 0$

(6) Pair of reaction at smooth contact with or without slip since $\dot{W} = N_1\bar{n}\cdot\bar{v}_{P_1} + (-N_1\bar{n})\cdot\bar{v}_{P_2} = N_1(\bar{n}\cdot\bar{v}_{P_1} - \bar{n}\cdot\bar{v}_{P_2}) = 0$

$\bar{v}_{P_1} = \bar{v}_{P_2}$

7. Pair of reaction $\bar{R}_1$ and $-\bar{R}_1$ at contact P_1 and P_2 with no slip since $\dot{W} = R_1\cdot v_{P_1} + (-R_1)\cdot v_{P_2} = 0$, $\bar{v}_{P_1} = \bar{v}_{P_2}$

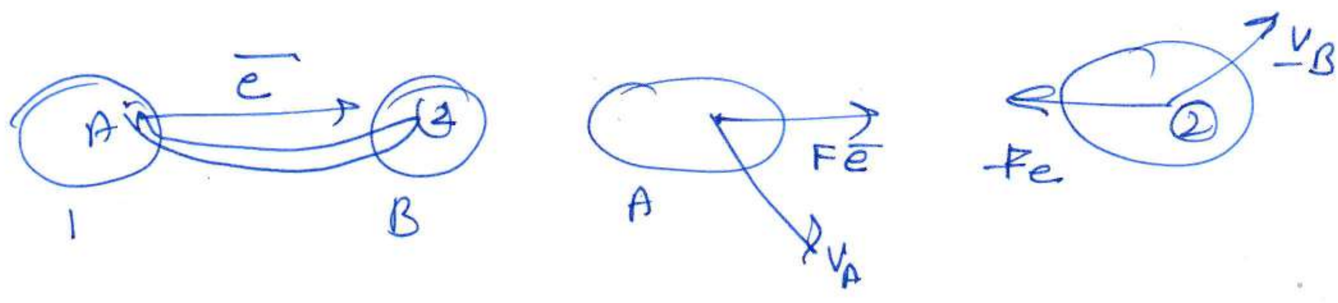
(8) Pair of tensions in light inextensible cable since $\dot{W} = T\bar{e}\cdot\bar{v}_1 + (-T\bar{e})\cdot\bar{v}_2 = T(\bar{e}\cdot\bar{v}_1 - \bar{e}\cdot\bar{v}_2) = 0$ as either $T=0$ for impending slackness or $e\cdot v_1 = e\cdot v_2$ for tight inextensible cable



(9) Pair of reactions due to light rigid rod with smooth ball and socket joint or (smooth hinge joints and coplanar load case)

since $\dot{W} = F\bar{e}\cdot\bar{v}_A + (-F\bar{e})\bar{v}_B = F\bar{e}\cdot(\bar{v}_A - \bar{v}_B)$

$\Rightarrow F\bar{e}\cdot\bar{\omega}\times\overline{AB} = F\bar{e}\cdot\bar{\omega}\times(AB)\bar{e} = 0$



Work Energy Relations for Interconnected System of Rigid Bodies.

$$\boxed{\begin{aligned} \dot{T} = \dot{W}, \quad & T_2 - T_1 = W_{1-2} \\ (T_2 + V_2) - (T_1 + V_1) &= W_{nc-12} \end{aligned}} \quad \text{(A)}$$

These equations are applied to individual bodies and added up to yield relations for the whole system which involve work done by the internal and the external forces on the system since some of the forces which are external and internal by the subscript int + ext, the work energy relation corresponding to reduce to the following form for varies cases:

$$\dot{T} = \dot{W}_{int+ext}, \quad T_2 - T_1 = W_{int+ext\,1-2}$$

$$\dot{T} + \dot{V}_{int+ext} = \dot{W}_{nc\,int+ext}, \quad (T_2 + V_{int+ext_2}) - (T_1 + V_{int+ext_1}) = W_{nc\,int+ext\,1-2}$$

$$\dot{T} + \dot{V}_{int+ext} = 0 \quad T_2 + V_{int+ext\,2} = T_1 + V_{int+ext\,1}$$

(B)

if all internal forces of interaction are together workless Then the equation (B) will reduce to eq. (A)