3-D Dynamics of Rigid Bodies

Introduction of third dimension

:: Third component of vectors representing force, linear velocity, linear acceleration, and linear momentum

:: Two additional components for vectors representing angular quantities: moments of forces, angular velocity, angular acceleration, and angular momentum
Example :: Flight simulator
Example :: Game of bowling
3-D Kinematics of Rigid Bodies

Several possible motions

Translation

:: Rectilinear Translation
  - Any two points in the body will move along parallel straight lines

:: Curvilinear Translation
  - Any two points in the body will move along congruent curves

:: Line joining the two points will always remain parallel to its original position

\[ \mathbf{r}_A = \mathbf{r}_B + \mathbf{r}_{A/B} \quad \mathbf{v}_A = \mathbf{v}_B \quad \mathbf{a}_A = \mathbf{a}_B \]

\( \mathbf{r}_{A/B} \) is constant and its time derivative is zero

:: All points in the body have same velocity and same acceleration
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Fixed Axis Rotation
:: $\omega$ does not change its direction since its lies along the fixed axis
:: Any point $A$ in the body (not on the axis) moves in a circular arc in a plane normal to the axis; its velocity:

$$v = \omega \times r$$

since $r = h + b$ and $\omega \times h = 0$

And acceleration of $A$ is given by time derivative:

$$a = \dot{\omega} \times r + \omega \times (\omega \times r)$$

since $\dot{r} = v = \omega \times r$

The normal and tangential components of $a$ for the circular motion:

$$a_n = |\omega \times (\omega \times r)| = b\omega^2$$
$$a_t = |\dot{\omega} \times r| = b\alpha$$

Since $v$ and $a$ are perpendicular to $\omega$ and $\dot{\omega}$:

$$v \cdot \omega = 0, v \cdot \dot{\omega} = 0, a \cdot \omega = 0, \text{ and } a \cdot \dot{\omega} = 0$$
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Parallel-Plane Motion

:: All points in a rigid body move in planes parallel to a fixed plane \( P \) known as plane of motion.

→ General form of Plane Motion

:: Motion of each point in the body, e.g., \( A' \), identical with the motion of the corresponding point \( A \) in the plane \( P \)

:: Kinematics of plane motion can be conveniently used

:: Principles of relative motion
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Rotation about a Fixed Point
:: Change in direction of angular velocity vector → more general rotation

Proper Vectors
:: Rotation vectors obeying the parallelogram law of vector addition → treated as proper vectors

Consider a solid sphere cut from a rigid body confined to rotate @ fixed point $O$

:: x-y-z axes are fixed and do not rotate with the body

:: Consider $90^\circ$ rotations of sphere
   @ x and y axes in two cases
   1. @ x- then @ y-axis
   2. @ y- then @ x-axis
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• Rotation about a Fixed Point

:: The two cases do not give the same final position
:: Finite rotations do not generally obey the parallelogram law of vector addition → not commutative
   → Finite rotations may not be treated as proper vectors
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Rotation about a Fixed Point

Infinitesimal Rotation

Combined effect of two infinitesimal rotations $d\theta_1$ and $d\theta_2$ of a rigid body @ respective axes through fixed point $O$

Displacement of $A$ due to $d\theta_1$: $d\theta_1 \times r$
Displacement of $A$ due to $d\theta_2$: $d\theta_2 \times r$

Same resultant displacement due to either order of addition of these infinitesimal displacements:
→ $d\theta_1 \times r + d\theta_2 \times r = (d\theta_1 + d\theta_2) \times r = d\theta \times r$
→ Two rotations equivalent to a single rotation $d\theta = d\theta_1 + d\theta_2$
→ Addition of angular velocities $\omega_1 = \dot{\theta}_1$ and $\omega_2 = \dot{\theta}_2$ gives: $\omega = \dot{\theta} = \omega_1 + \omega_2$
→ Infinitesimal rotations obey the parallelogram law of vector addition

→ At any instant of time, a body with one fixed point rotates instantaneously @ a particular axis passing through the fixed point
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Rotation about a Fixed Point

Instantaneous Axis of Rotation

:: Rotation of solid rotor with black particles embedded on its surface

:: $\omega_1$ and $\omega_2$ are steady angular velocities

:: Along line $O-n$, velocity of particles will be momentarily zero (the line will be sharply defined)

:: The line of points with no velocity

$\rightarrow$ Instantaneous position of the Axis of Rotation

:: Any point on this line, e.g., $A$, would have equal and opposite velocity components $v_1$ and $v_2$ due to $\omega_1$ and $\omega_2$, respectively

All particles of the body, except those on line $O-n$, rotate momentarily in circular arcs @ the instantaneous axis of rotation
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Rotation about a Fixed Point

Instantaneous Axis of Rotation

:: Change of position of rotation axis both in space and relative to the body

Body and Space Cones

:: Relative to the cylinder, the instantaneous axis of rotation $O-A-n$ generates a right circular cone @ the cylinder axis $O \rightarrow$ Body Cone

:: Instantaneous axis of rotn also generates a right circular cone @ the vertical axis $\rightarrow$ Space Cone

:: Rolling of body cone on space cone

:: Angular vel $\omega$ of the body located along the common element of the two cones
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Rotation about a Fixed Point

Angular Acceleration

:: Planar motion
   - scalar $\alpha$ reflects the change in magnitude of $\omega$

:: 3-D motion
   - vector $\alpha$ reflects the change in the direction and magnitude of $\omega$

:: $\alpha$ - tangent vector to the space curve $p$ in the direction of the change in $\omega$

Vel of a point on a rigid body
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Rotation about a Fixed Point

If the magnitude of $\omega$ remains constant, $\alpha$ is normal to $\omega$

Let $\Omega$ be the angular velocity with which the vector $\omega$ itself rotates (precesses) as it forms the space cone

$\alpha = \Omega \times \omega$

:: $\alpha$, $\Omega$, and $\omega$ have the same relationship as that of $v$, $\omega$, and $r$ defining the vel of a point $A$ in a rigid body
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Rotation about a Fixed Point
When a rigid body rotates @ a fixed point $O$ with the instantaneous axis of rotation $n-n$, vel $v$ and accln $a = \dot{v}$ of any point $A$ in the body are given by the same expressions derived when the axis was fixed:

$$v = \omega \times r$$

$$a = \dot{\omega} \times r + \omega \times (\omega \times r)$$

Rotation about a Fixed Axis vs Rotation about a Fixed Point

For rotation @ a fixed point:
- angular accln will have a comp normal to $\omega$ due to change in dirn of $\omega$, as well as a comp in the dirn of $\omega$ to reflect any change in the magnitude of $\omega$
- Although any point on the rotation axis $n-n$ momentarily will have zero velocity, it will not have zero accln as long as $\omega$ is changing its direction

For rotation @ a fixed axis:
- angular accln will have only one component along the fixed axis to reflect any change in the magnitude of $\omega$.
- Points which lie on the fixed axis of rotation will have no velocity or accln
Example (1) on 3-D Kinematics

The 0.8-m arm OA for a remote-control mechanism is pivoted about the horizontal x-axis of the clevis, and the entire assembly rotates about the z-axis with a constant speed $N = 60$ rev/min. Simultaneously, the arm is being raised at the constant rate $\beta = 4$ rad/s. For the position where $\beta = 30^\circ$, determine (a) the angular velocity of OA, (b) the angular acceleration of OA, (c) the velocity of point A, and (d) the acceleration of point A. If, in addition to the motion described, the vertical shaft and point O had a linear motion, say, in the z-direction, would that motion change the angular velocity or angular acceleration of OA?
Example (1) on 3-D Kinematics

Solution:
(a) Since the arm is rotating @ both $x$ and $z$ axis, it has the Angular velocity components:

$$\omega_x = \dot{\beta} = 4 \text{ rad/s} \quad \text{and} \quad \omega_z = \frac{2\pi N}{60} = \frac{2\pi (60)}{60} = 6.28 \text{ rad/s}$$

Angular vel of $OA$:

$$\omega = \omega_x + \omega_z = 4\mathbf{i} + 6.28\mathbf{k} \text{ rad/s}$$

(b) Angular accln of $OA$:

$\omega_z$ is not changing in magnitude or direction $\Rightarrow \omega_z = 0$

$\omega_x$ is changing direction $\Rightarrow$ has a derivative

Using $\alpha = \Omega \times \omega$

$$\dot{\omega}_x = \omega_z \times \omega_x = 6.28\mathbf{k} \times 4\mathbf{i} = 25.1\mathbf{j} \text{ rad/s}^2$$

$$\Rightarrow \alpha = 25.1\mathbf{j} + 0 = 25.1\mathbf{j} \text{ rad/s}^2$$

Directions of angular vel and accln are shown in the vector diagram.
Example (1) on 3-D Kinematics

c) For determining linear vel and accln of point A at the given instant, the problem may be considered as a fixed axis rotation problem.

Position vector of A: \( \mathbf{r} = 0.693\mathbf{j} + 0.4\mathbf{k} \text{ m} \)

Vel of A:

\[
\mathbf{v} = \omega \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 0 & 6.28 \\ 0 & 0.693 & 0.4 \end{vmatrix} = -4.35\mathbf{i} - 1.60\mathbf{j} + 2.77\mathbf{k} \text{ m/s}
\]
Example (1) on 3-D Kinematics

(d) Acceleration of A:

\[
a = \dot{\omega} \times r + \omega \times (\omega \times r)
\]

\[
\begin{array}{ccc}
  i & j & k \\
 0 & 25.1 & 0 \\
0 & 0.693 & 0.4
\end{array}
\] +

\[
\begin{array}{ccc}
  i & j & k \\
 4 & 0 & 6.28 \\
-4.35 & -1.60 & 2.77
\end{array}
\]

= \(10.05i\) + \(10.05i - 38.4j - 6.40k\)

= \(20.1i - 38.4j - 6.40k\) m/s^2

Angular motion of OA depends only on the angular changes \(N\) and \(\dot{\beta}\).

→ Any linear motion of O does not affect \(\omega\) and \(\alpha\).
Example (2) on 3-D Kinematics

The electric motor with an attached disk is running at a constant low speed of 120 rev/min in the direction shown. Its housing and mounting base are initially at rest. The entire assembly is next set in rotation about the vertical Z-axis at the constant rate $N = 60$ rev/min with a fixed angle $\gamma$ of 30°. Determine (a) the angular velocity and angular acceleration of the disk, (b) the space and body cones, and (c) the velocity and acceleration of point A at the top of the disk for the instant shown.

Solution

Axes $x$, $y$, $z$ with unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are attached to the motor frame.

- $x$-axis coincides with the horz axis through O @ which the motor tilts.
- $z$-axis coincides with the rotor axis
- $y$-axis will be as shown normal to $x$ and $z$
- $Z$-axis is vertical and has unit vectors:
  \[ \mathbf{K} = j\cos\gamma + k\sin\gamma \]
Example (2) on 3-D Kinematics

Solution: \( \mathbf{K} = \mathbf{j} \cos \gamma + \mathbf{k} \sin \gamma \)

(a) The rotor and disk have two components of angular velocity:

- @ z-axis: \( \omega_o = 120(2\pi)/60 = 4\pi \text{ rad/s} \)
- @ Z-axis: \( \Omega = 60(2\pi)/60 = 2\pi \text{ rad/s} \)

Angular velocity of the disk:

\[
\omega = \omega_0 + \Omega = \omega_0 \mathbf{k} + \Omega \mathbf{K}
\]

\[
= \omega_0 \mathbf{k} + \Omega (\mathbf{j} \cos \gamma + \mathbf{k} \sin \gamma) = (\Omega \cos \gamma)\mathbf{j} + (\omega_0 + \Omega \sin \gamma)\mathbf{k}
\]

\[
= (2\pi \cos 30^\circ)\mathbf{j} + (4\pi + 2\pi \sin 30^\circ)\mathbf{k} = \pi(\sqrt{3}\mathbf{j} + 5.0\mathbf{k}) \text{ rad/sec}
\]

It is clear from the vector diagram that

\( \omega = \omega_o + \Omega = \omega_y + \omega_z \)
Example (2) on 3-D Kinematics

(a) Since the magnitude of the angular vel $\omega$ is constant (steady precession), Angular acceleration of the disk:

$$\alpha = \dot{\omega} = \dot{\Omega} \times \omega$$

$$= \Omega (j \cos \gamma + k \sin \gamma) \times [(\Omega \cos \gamma)j + (\omega_0 + \Omega \sin \gamma)k]$$

$$= \Omega (\omega_0 \cos \gamma + \Omega \sin \gamma \cos \gamma) i - (\Omega^2 \sin \gamma \cos \gamma) i$$

$$= (\Omega \omega_0 \cos \gamma) i = i (2\pi)(4\pi) \cos 30^\circ = 68.4 i \text{ rad/sec}^2$$

(b) Space Cone and Body Cone:

Angular velocity vector $\omega$ is the common element of the space and body cones.

Since magnitude of the angular vel $\omega$ is constant, angular accln $\alpha$ must be tangent to the base circle of the space cone and $\alpha$ must be normal to $\omega$

$\Rightarrow \alpha$ must be along $x$-direction as calculated.
Example (2) on 3-D Kinematics

Solution: \( K = j \cos \gamma + k \sin \gamma \)

(c) Linear instantaneous Vel and Accln of A
- Rotation @ fixed axis can be considered because axis of rotation is fixed at the instant \( \gamma = 30^\circ \)

\[ \mathbf{v} = \mathbf{\omega} \times \mathbf{r} \]
\[ \mathbf{a} = \mathbf{\dot{\omega}} \times \mathbf{r} + \mathbf{\omega} \times (\mathbf{\omega} \times \mathbf{r}) \]

Position vector of A for the instant considered: \( \mathbf{r} = 0.125\mathbf{j} + 0.250\mathbf{k} \) m

Velocity of point A = \( \mathbf{\omega} \times \mathbf{r} \)

\[
\mathbf{v} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
0 & \sqrt{3}\pi & 5\pi \\
0 & 0.125 & 0.250
\end{vmatrix} = -0.6\mathbf{i} \text{ m/s}
\]

Accln of point A:

\[
\mathbf{a} = \mathbf{\ddot{\omega}} \times \mathbf{r} + \mathbf{\omega} \times (\mathbf{\omega} \times \mathbf{r}) = \mathbf{\alpha} \times \mathbf{r} + \mathbf{\omega} \times \mathbf{v}
\]

\[
= 68.4\mathbf{i} \times (0.125\mathbf{j} + 0.250\mathbf{k}) + \pi(\sqrt{3}\mathbf{j} + 5\mathbf{k}) \times (-0.6\mathbf{i})
\]

\[
\mathbf{a} = -26.6\mathbf{j} + 11.83\mathbf{k} \text{ m/s}^2
\]