Kinetics of Particles: Relative Motion

D’Alembert’s Principle

- Accln of a particle measured from fixed set of axes X-Y-Z is its absolute acceleration \((a)\).
  - Newton’s second law of motion can be applied \((\Sigma F = ma)\)
- If the particle is observed from a moving system \((x-y-z)\) attached to a particle, the particle appears to be at rest or in equilibrium in \(x-y-z\).
  - Therefore, the observer who is accelerating with \(x-y-z\) would conclude that a force \(-ma\) acts on the particle to balance \(\Sigma F\).
  - Treatment of dynamics problem by the method of statics \(\rightarrow\) work of D’Alembert (1743)
  - As per this approach, Equation of Motion is rewritten as:
    - \(\Sigma F - ma = 0\) \(\rightarrow\) \(-ma\) is also treated as a force
    - This fictitious force is known as Inertia Force
    - The artificial state of equilibrium created is known as Dynamic Equilibrium.
  - Transformation of a problem in dynamic to one in statics is known as D’Alembert’s Principle.
Kinetics of Particles: Potential Energy

Conservative Force Fields

• Work done against a gravitational or elastic force depends only on net change of position and not on the particular path followed in reaching the new position.
• Forces with this characteristic are associated with Conservative Force Fields
  → A conservative force is a force with the property that the work done by it in moving a particle between two points is independent of the path taken
  (Ex: Gravity is a conservative force, friction is a non-conservative force)
  → these forces possess an important mathematical property.

Consider a force field where the force $\mathbf{F}$ is a function of the coordinates. Work done by $\mathbf{F}$ during displacement $d\mathbf{r}$ of its point of application: $dU = \mathbf{F} \cdot d\mathbf{r}$

→ Total work done along its path from 1 to 2:

$$U = \int \mathbf{F} \cdot d\mathbf{r} = \int (F_x \, dx + F_y \, dy + F_z \, dz)$$

• In general, the integral $\int \mathbf{F} \cdot d\mathbf{r}$ is a line integral that depends on the path followed between 1 and 2.
• For some forces, $\mathbf{F} \cdot d\mathbf{r}$ is an exact differential $-dV$ of some scalar function $V$ of the coordinates (minus sign for $dV$ is arbitrary but agree with the sign of PE change in the gravity field of the earth)
Kinetics of Particles: Potential Energy

Conservative Force Fields
If \( \mathbf{F} \cdot d\mathbf{r} \) is an exact differential \(-dV\) of some scalar function \( V \) of the coordinates:

\[
U_{1,2} = \int_{V_1}^{V_2} -dV = -(V_2 - V_1)
\]

\( \rightarrow \) The work done depends on only the end points of the motion (i.e., independent of the path followed!)

If \( V \) exists, differential change in \( V \) becomes:

\[
dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz
\]

Using

\[
-dV = \mathbf{F} \cdot d\mathbf{r} = F_x \, dx + F_y \, dy + F_z \, dz
\]

\( \rightarrow \)
\[
F_x = -\frac{\partial V}{\partial x}, \quad F_y = -\frac{\partial V}{\partial y}, \quad F_z = -\frac{\partial V}{\partial z}
\]

For any closed path (points 1 and 2 coincide), work done by the conservative force field \( \mathbf{F} \) is zero:

\[
\oint \mathbf{F} \cdot d\mathbf{r} = 0
\]

(circle on the integration sign indicates that the path is closed)

The force may also be written as the vector: \( \mathbf{F} = -\nabla V \) or

\[
\mathbf{F} = -\frac{\partial V}{\partial x} \mathbf{i} - \frac{\partial V}{\partial y} \mathbf{j} - \frac{\partial V}{\partial z} \mathbf{k}
\]

The vector operator “del”:

\[
\nabla = \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z}
\]

\( V \) is known as the Potential Function and the expression \( \nabla V \) is known as the gradient of the potential function \( V \) (scalar)

\( \rightarrow \)

\[
\nabla \times \mathbf{F} = -\nabla \times \Delta V = 0
\]

Since curl of the gradient of any scalar function is a zero vector.
Kinetics of Particles: Potential Energy

Conservative Force Fields
A force field \( F \) is said to be conservative if it meets any of the following three equivalent conditions:

1. \[ F = -\nabla V \]
   \[ \rightarrow \] If the force can be written as the negative gradient of a potential. When force components are derivable from a potential, the force is said to be conservative, and the work done by the force between any two points is independent of the path followed.

2. \[ \oint F \cdot dr = 0 \]
   \[ \rightarrow \] For any closed path, net work done by the conservative force field is zero.

3. \[ \nabla \times F = 0 \]
   \[ \rightarrow \] If the curl of the force is zero.
Kinetics of Systems of Particles

Generalized Newton’s Second Law

• $n$ mass particles bounded by a closed surface in space

• $F_1, F_2, F_3, \ldots$ acting on $m_i$ from sources external to the envelope
  :: e.g., contact with external bodies, gravitational, electric, magnetic etc.)

• $f_1, f_2, f_3, \ldots$ acting on $m_i$ from sources internal to the system boundary
  :: reaction forces from other mass particles within the boundary

• Mass centre $G$ can be located by $\overline{m\bar{r}} = \Sigma m_i \bar{r}_i$
Kinetics of Systems of Particles

Generalized Newton’s Second Law

Applying Newton’s Second law to \( m_i \):

\[
F_1 + F_2 + F_3 + \cdots + f_1 + f_2 + f_3 + \cdots = m_i \ddot{r}_i
\]

For all particles of the system:

\[
\sum F + \sum f = \sum m_i \ddot{r}_i
\]

\( \sum f = 0 \) since all internal forces occur in pairs of equal and opposite actions and reactions.

\[
\sum F = \sum m_i \ddot{r}_i
\]

(assuming constant \( m \))

\( \ddot{a} \) is the accln of CM (has same direction as \( \sum F \))

\( \sum F \) does not necessarily pass through \( G \)

\[
\sum F_x = m \ddot{a}_x \quad \sum F_y = m \ddot{a}_y \quad \sum F_z = m \ddot{a}_z
\]

\( \rightarrow \) Generalized Law in component form
Kinetics of Systems of Particles

Work-Energy

• Work-energy relation for a particle:
  \[ U_{1-2} = T_2 - T_1 = \Delta T \]

• Work-energy relation for \( m_i \):
  \[ (U_{1-2})_i = \Delta T_i \]

\( (U_{1-2})_i \) is work done by \( m_i \) during an interval of motion by all forces (external \( F_1 + F_2 + \ldots \) and internal \( f_1 + f_2 + \ldots \))

• KE of \( m_i \): \( T_i = \frac{1}{2} m_i v_i^2 \)
  \[ v_i = \dot{r}_i \]
  \( v_i \) is the magnitude of the particle velocity

For the entire system: \( \sum (U_{1-2})_i = \sum \Delta T_i \)

\( \rightarrow \) Same work-energy relation:
  \[ U_{1-2} = \Delta T \] or \[ T_1 + U_{1-2} = T_2 \]
Kinetics of Systems of Particles

• Work-Energy

\[ U_{1-2} = \sum (U_{1-2})_i \] is total work done by all forces on all particles.

\[ \Delta T \] is the change in the total KE, \( T = \sum T_i \), of the system

\[ \text{::} \] In rigid bodies, work done by all pairs of internal forces is zero
\[ \rightarrow U_{1-2} \] is the total work done by only the external forces on the system.

\[ \text{::} \] In non-rigid systems, conversion into change in the internal elastic PE \( V_e \).

\[ U'_{1-2} = \Delta T + \Delta V \]

\[ T_1 + V_1 + U'_{1-2} = T_2 + V_2 \]

\[ V = V_e + V_g = \text{Total PE} \]
Kinetics of Systems of Particles

Work-Energy

KE of the mass system: \( T = \sum \frac{1}{2} m_i v_i^2 \)

Relative Motion:

\[ v_i = \bar{v} + \rho_i \]

\[ v_i^2 = \mathbf{v}_i \cdot \mathbf{v}_i \]

KE of the system:

\[
T = \sum \frac{1}{2} m_i \mathbf{v}_i \cdot \mathbf{v}_i = \sum \frac{1}{2} m_i (\mathbf{v} + \dot{\rho}_i) \cdot (\mathbf{v} + \dot{\rho}_i)
\]

\[ = \sum \frac{1}{2} m_i \bar{v}^2 + \sum \frac{1}{2} m_i |\dot{\rho}_i|^2 + \Sigma m_i \mathbf{v} \cdot \dot{\rho}_i \]

Since \( \rho_i \) is measured from the mass center

\[ \therefore \text{The third term:} \quad \mathbf{v} \cdot \Sigma m_i \dot{\rho}_i = \mathbf{v} \cdot \frac{d}{dt} \Sigma (m_i \rho_i) = 0 \]

and

\[ \Sigma \frac{1}{2} m_i \bar{v}^2 = \frac{1}{2} \bar{v}^2 \Sigma m_i = \frac{1}{2} m \bar{v}^2 \]

Therefore, the total KE:

\[ T = \frac{1}{2} m \bar{v}^2 + \Sigma \frac{1}{2} m_i |\dot{\rho}_i|^2 \]

Total KE of a mass system = KE of the mass center translation of the system as a whole + KE due to motion of all particles relative to the mass center
**Kinetics of Systems of Particles**

**Impulse-Momentum**

**Linear Momentum** (for single particle: \( G = mv \))

\[
G_i = m_i v_i \quad \dot{v}_i = \dot{r}_i
\]

Linear Momentum of the system is defined as the vector sum of the LM of all its particles

\[
G = \sum m_i v_i
\]

Now

\[
v_i = \bar{v} + \dot{\rho}_i \quad \sum m_i \dot{\rho}_i = 0
\]

\[
\Rightarrow \quad G = \sum m_i (\bar{v} + \dot{\rho}_i) = \sum m_i \bar{v} + \frac{d}{dt} \sum m_i \rho_i = \bar{v} \sum m_i + \frac{d}{dt} (0)
\]

\[
\Rightarrow \quad G = m \bar{v}
\]

**LM of any system of constant mass** is the **product** of the **mass** and the **velocity** of its center of mass

Time derivative of \( G \):

\[
m \bar{v} = m \ddot{a} = \sum F
\]

\[
\Rightarrow \quad \sum F = \dot{G}
\]

→ Has the same form as that for a single particle. **Resultant** of the **external forces** on **any mass system** = time rate of change of LM of the system
Kinetics of Systems of Particles

**Impulse-Momentum**

**Angular Momentum**

AM of a single particle: \( H_O = r \times mv \)

AM of general mass system @

- **Fixed Point** \( O \)
- **Mass Center** \( G \)
- **An arbitrary point** \( P \) with accln

\[
a_P = \ddot{r}_P
\]

**Angular Momentum @ a fixed point** \( O \)

AM of the mass system @ the point \( O \) fixed in the Newtonian Reference System = the **vector sum of the moments** of the LM @ \( O \) of all particles of the system

\[
H_O = \Sigma (r_i \times m_i \mathbf{v}_i)
\]
Impulse-Momentum
Angular Momentum

The time derivative of the vector product:

\[ \dot{H}_O = \sum (\dot{r}_i \times m_i \dot{v}_i) + \sum (r_i \times m_i \ddot{v}_i) \]

First term represents cross product of two parallel vectors \( \rightarrow \) zero

The second term:

\[ \sum (r_i \times m_i \ddot{v}_i) = \sum (r_i \times F_i) \]

\( \rightarrow = \sum \mathbf{M}_O \) that represents only moments of the external forces

\[ \sum \mathbf{M}_O = \dot{H}_O \]

\( \rightarrow \) Similar to the eqn for a single particle. Resultant vector moment of all ext forces @ any fixed point = time rate of change of AM of the system @ the fixed point. (Eqn can also be applied to non-rigid systems)
Kinetics of Systems of Particles

**Impulse-Momentum**

**Angular Momentum**

Angular Momentum @ mass Center $G$

AM of the mass system @ $G = \text{sum of the moments of the LM @ } G$
of all particles of the system

$H_G = \sum \rho_i \times m_i \dot{r}_i$

We may write the absolute velocity

$H_G = \sum \rho_i \times m_i (\ddot{r} + \dot{\rho}_i) = \sum \rho_i \times m_i \ddot{r} + \sum \rho_i \times m_i \dot{\rho}_i$

$H_G = \sum \rho_i \times m_i \dot{\rho}_i$

Relative AM because relative velocity is used

If $G$ is taken as the reference, the absolute and relative AM will be identical

Differentiating first eqn wrt time:

$H_G = \sum \dot{\rho}_i \times m_i (\ddot{r} + \dot{\rho}_i) + \sum \rho_i \times m_i \ddot{r}_i$

First term:

$\sum \dot{\rho}_i \times m_i \ddot{r} + \sum \dot{\rho}_i \times m_i \dot{\rho}_i$

This is zero because both the terms are zero since the first term may be written as:
Kinetics of Systems of Particles

Impulse-Momentum

Angular Momentum

Angular Momentum @ mass Center $G$

\[ \dot{H}_G = \sum \dot{\rho}_i \times m_i (\ddot{\mathbf{r}} + \dot{\rho}_i) + \sum \rho_i \times m_i \dot{\mathbf{r}}_i \]

The first term is seen to be zero.

Second term using Newton’s second law:

\[ \sum \rho_i \times (\mathbf{F}_i + \mathbf{f}_i) = \sum \rho_i \times \mathbf{F}_i = \sum \mathbf{M}_G \]

→ Sum of all external moments @ $G$

→ \[ \sum \mathbf{M}_G = \dot{H}_G \]

- We may use either the absolute or the relative momentum
- The eqn can also be applied to non-rigid systems
- Mass should remain constant
Kinetics of Systems of Particles

Impulse-Momentum

Angular Momentum

Angular Momentum @ an arbitrary point $P$

- Point $P$ may have an accln  $a_p = \ddot{r}_P$

$$H_P = \sum \rho'_i \times m_i \dot{r}_i = \sum (\rho + \rho_i) \times m_i \dot{r}_i$$

The first term:  $\rho \times \sum m_i \dot{r}_i = \rho \times \sum m_i \dot{v}_i = \rho \times m \dot{v}$

The second term:  $\sum \rho_i \times m_i \dot{r}_i = H_G$

$\Rightarrow H_P = H_G + \rho \times m \dot{v}$  

Absolute AM @ any point $P = AM @ G +$ moment @ $P$ of the LM of the system considered concentrated at $G$

Using the Principle of Moments

Resultants of the ext forces acting on the system can be expressed as resultant force $\sum F$ through $G$ and the corresponding couple $\sum M_G$

$\sum M_P = \sum M_G + \rho \times \sum F$

$\Rightarrow \sum M_P = \dot{H}_G + \rho \times m \ddot{a}$
Kinetics of Systems of Particles

**Impulse-Momentum**

**Angular Momentum**

Angular Momentum @ an arbitrary point \( P \)

Using the Principle of Moments

\[
\Sigma M_P = \dot{H}_G + \bar{\rho} \times m\ddot{a}
\]

Similar eqn can also be developed using momentum relative to \( P \):

\[
(H_P)_\text{rel} = \Sigma \rho_i' \times m_i \dot{\rho}_i'
\]

Substituting \( \rho_i' = \bar{\rho} + \rho_i \) and \( \dot{\rho}_i' = \dot{\rho} + \dot{\rho}_i \)

\[
(H_P)_\text{rel} = \Sigma \bar{\rho} \times m_i \dot{\rho} + \Sigma \rho \times m_i \dot{\rho}_i + \Sigma \rho_i \times m_i \dot{\rho} + \Sigma \rho_i \times m_i \dot{\rho}_i
\]

The first term:

\[
\bar{\rho} \times m \bar{v}_\text{rel}
\]

Second will be zero:

Third term will be zero:

Fourth term is: \((H_G)_\text{rel}\) which is same as \(H_G\)

\[
(H_P)_\text{rel} = (H_G)_\text{rel} + \bar{\rho} \times m \bar{v}_\text{rel}
\]
Kinetics of Systems of Particles

Impulse-Momentum

Angular Momentum

Angular Momentum @ an arbitrary point $P$

Using the Principle of Moments

$$(H_P)_{\text{rel}} = (H_G)_{\text{rel}} + \rho \times m\overline{v}_{\text{rel}}$$

Differentiating defn $$(H_P)_{\text{rel}} = \sum \rho_i' \times m_i \dot{\rho}_i'$$ wrt time and substituting $\dot{\mathbf{r}}_i = \dot{\mathbf{r}}_P + \dot{\rho}_i'$$

$$(\dot{H}_P)_{\text{rel}} = \sum \rho_i' \times m_i \dot{\rho}_i' + \sum \rho_i' \times m_i \ddot{r}_i - \sum \rho_i' \times m_i \ddot{r}_P$$

First term is zero. Second term is $\sum M_P$ (moment of ext $F$)

Third term:

$$\sum \rho_i' \times m_i \mathbf{a}_P = -\mathbf{a}_P \times \sum m_i \rho_i' = -\mathbf{a}_P \times m\overline{\rho} = \overline{\rho} \times m \mathbf{a}_P$$

$\sum M_P = (\dot{H}_P)_{\text{rel}} + \overline{\rho} \times m \mathbf{a}_P$

Convenient when a point $P$ whose accln is known is used as a moment center

$\sum M_P = (\dot{H}_P)_{\text{rel}}$

The eqn reduces to a simpler form

If (1) $\mathbf{a}_P = 0$ equivalent to first case (AM @ $O$)

(2) $\overline{\rho} = 0$ equivalent to second case (AM @ $G$)

(3) $\overline{\rho}$ and $\mathbf{a}_P$ are parallel ($\mathbf{a}_P$ directed toward or away from $G$)
Kinetics of Systems of Particles

Conservation of Energy
• A mass system is said to be conservative if it does not lose energy by virtue of internal friction forces that do negative work or by virtue of inelastic members that dissipate energy upon cycling.
• If no work is done on a conservative system during an interval of motion by external forces other than gravity or other potential forces, then energy of the system is not lost \( U'_{1-2} = 0 \)

\[
\Delta T + \Delta V = 0 \quad \text{or} \quad T_1 + V_1 = T_2 + V_2
\]

Conservation of Momentum
In absence of external impulse (resultant external force zero)

\( G_1 = G_2 \)

In absence of external angular impulse (resultant moment @ \( G \) or \( O \) of all external forces zero)

\[
(H_O)_1 = (H_O)_2 \quad \text{or} \quad (H_G)_1 = (H_G)_2
\]
Plane Kinematics of Rigid Bodies

Rigid Body
• A system of particles for which the distances between the particles remain unchanged.
• This is an ideal case. There is always some deformation in materials under the action of loads. This deformation can be neglected if the changes in the shape are small compared to the movement of the body as a whole.

Particle Kinematics
• Developed the relationships governing the disp, vel, and accln of points as they move along straight or curved paths.

Rigid Body Kinematics
• Same relationships will be used. In addition, rotational motion of rigid bodies will also be accounted for.
• Involves both linear and angular disp, vel, and accln.
• Discussion will be restricted to motion in a Single Plane (Plane Motion).
  ▪ When all parts of the body move in parallel planes
  ▪ Plane containing mass center is generally considered as plane of motion and the body is treated as a thin slab whose motion is confined to the plane of the slab.
## Plane Kinematics of Rigid Bodies

### Plane Motion

#### Translation
No rotation of any line in body. Motion of the body specified by motion of any point in the body \( \approx \) Motion of a single particle.

#### Rotation @ a Fixed Axis
All particles move in circular paths @ axis of rotn. All lines perpendicular to the axis of rotn rotate through the same angle.

### General Planar Motion
Combination of translation and rotation

The actual paths of all particles in the body are projected on to a single plane of motion.

<table>
<thead>
<tr>
<th>Type of Rigid-Body Plane Motion</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) Rectilinear translation</td>
<td>Rocket test sled</td>
</tr>
<tr>
<td>(b) Curvilinear translation</td>
<td>Parallel-link swinging plate</td>
</tr>
<tr>
<td>(c) Fixed-axis rotation</td>
<td>Compound pendulum</td>
</tr>
<tr>
<td>(d) General plane motion</td>
<td>Connecting rod in a reciprocating engine</td>
</tr>
</tbody>
</table>
Plane Kinematics of Rigid Bodies

Rotation

- Described by angular motion
Consider plane motion of a rotating rigid body
\[ \theta_2 = \theta_1 + \beta \]
since \( \beta \) is invariant
Therefore, \( \dot{\theta}_2 = \dot{\theta}_1 \) and \( \ddot{\theta}_2 = \ddot{\theta}_1 \)

And, during a finite interval: \( \Delta \theta_2 = \Delta \theta_1 \)

-> All lines on a rigid body in its plane of motion have the same angular displacement, same angular velocity (\( \omega \)), and same angular acceleration (\( \alpha \)).

For plane Rotation:

\[ \omega = \frac{d\theta}{dt} = \dot{\theta} \]
\[ \alpha = \frac{d\omega}{dt} = \dot{\omega} \]
\[ \omega \, d\omega = \alpha \, d\theta \]

+ve dirn for \( \omega \) and \( \alpha \) is the same as that chosen for \( \theta \)

Particle Kinematics \( \rightarrow \) Rigid Body Kinematics

\( (s, v, a) \) \( \rightarrow \) \( (\theta, \omega, \alpha) \)

For rotation with constant angular acceleration, integrals of these eqns give:

\[ \omega = \omega_0 + \alpha t \]
\[ \omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0) \]
\[ \theta = \theta_0 + \omega_0 t + \frac{1}{2} \alpha t^2 \]

At \( t = 0 \), \( \theta = \theta_0 \), \( \omega = \omega_0 \)
Plane Kinematics of Rigid Bodies

Rotation @ a Fixed Axis

- All points (except those on the axis) move in concentric circles @ the fixed axis normal to the plane of the fig.

Vel of A using $n-t$ for circular motion:

$v_n = r = 0$, \[ \therefore v = v_t = r \dot{\theta} = r \omega \]

For particle circular motion

\[
\begin{align*}
  v &= r \dot{\theta} \\
  a_n &= v^2/r = r \dot{\theta}^2 = v \ddot{\theta} \\
  a_t &= \ddot{v} = r \dddot{\theta}
\end{align*}
\]

For body rotational motion

\[
\begin{align*}
  v &= r \omega \\
  a_n &= r \omega^2 = v^2/r = v \omega \\
  a_t &= r \alpha
\end{align*}
\]

Alternatively, using the cross-product relationship of vector notation:

\[ \mathbf{v} = \dot{\mathbf{r}} = \mathbf{r} \times \mathbf{\omega} \]

the reverse order of vectors will give

\[ \mathbf{r} \times \mathbf{\omega} = -\mathbf{v} \]

\[ \mathbf{a} = \ddot{\mathbf{v}} = \mathbf{\omega} \times \mathbf{\dot{r}} + \dot{\mathbf{\omega}} \times \mathbf{r} = \mathbf{\omega} \times (\mathbf{\omega} \times \mathbf{r}) + \mathbf{\dot{\omega}} \times \mathbf{r} = \mathbf{\omega} \times \mathbf{v} + \mathbf{\alpha} \times \mathbf{r} \]

The vector equivalent eqns:

For 3-D motion, $\omega$ may change dirn as well as magnitude \[ \rightarrow \alpha \text{ and } \omega \text{ will have different dirns} \]
Plane Kinematics of Rigid Bodies

Absolute Motion Analysis

• First approach to rigid body kinematics

• Similar to the absolute motion analysis of the constrained motion of connected particles discussed in kinematics of particles (for pulley configurations considered, the relevant velocities and accelerations were obtained by successive differentiation of the lengths of the connecting cables).

• In case of absolute motion analysis of rigid bodies, both linear and angular velocities and linear and angular accelerations are required to be considered.
Plane Kinematics of Rigid Bodies

Absolute Motion Analysis

Example

A wheel of radius \( r \) rolls on a flat surface without slipping. Determine the angular motion of the wheel in terms of the linear motion of its center \( O \). Also determine the acceleration of a point on the rim of the wheel as the point comes into contact with the surface on which the wheel rolls.

Solution: Part A
\( O \rightarrow O' \)
\( OO' = s = \text{arc length } C'A \) (since no slippage)
Radial line \( CO \rightarrow C'O' @ \theta \) from vertical

\[
\begin{align*}
s &= r\theta \\
v_O &= r\omega \\
a_O &= r\alpha
\end{align*}
\]

\[
\begin{align*}
v_O &= \dot{s}, \quad a_O = \dot{v}_O = \ddot{s}, \\
\omega &= \dot{\theta}, \quad \alpha = \dot{\omega} = \dot{\theta}
\end{align*}
\]

The accln \( a_O \) and \( \alpha \) will have opposite sense to the velocity \( v_O \) and \( \omega \), respectively if the wheel is slowing down.
Plane Kinematics of Rigid Bodies

Absolute Motion Analysis

Example

Solution: Part B

When C has moved along its cycloidal path to C':
Its new coordinates and their time derivatives becomes:

\[
\begin{align*}
  x &= s - r \sin \theta = r(\theta - \sin \theta) \\
  \dot{x} &= r \dot{\theta} (1 - \cos \theta) = v_O (1 - \cos \theta) \\
  \ddot{x} &= \dot{v}_O (1 - \cos \theta) + v_O \dot{\theta} \sin \theta \\
  &= a_O (1 - \cos \theta) + r \omega^2 \sin \theta
\end{align*}
\]

\[
\begin{align*}
  y &= r - r \cos \theta = r(1 - \cos \theta) \\
  \dot{y} &= r \dot{\theta} \sin \theta = v_O \sin \theta \\
  \ddot{y} &= \dot{v}_O \sin \theta + v_O \dot{\theta} \cos \theta \\
  &= a_O \sin \theta + r \omega^2 \cos \theta
\end{align*}
\]

\(\theta\) is measured from vertical direction. When any point on rim of the wheel comes into contact with the supporting surface, \(\theta = 0\). Substituting in above eqns:

\[
\begin{align*}
  \dot{x} &= \dot{y} = 0 \\
  \ddot{x} &= 0 \\
  \dddot{y} &= r \omega^2
\end{align*}
\]

When \(\theta = 0\), the point of contact has zero velocity.

Also, accln of any point on rim (Ex C) at the instant of contact with the ground depends only on \(r\) and \(\omega\), and is directed toward the center of the wheel.

**Vel and accln of C at any position \(\theta\):**

\[
\mathbf{v} = \dot{x} \mathbf{i} + \dot{y} \mathbf{j} \quad \text{and} \quad \mathbf{a} = \ddot{x} \mathbf{i} + \dddot{y} \mathbf{j}
\]