Kinetics of Particles: Work and Energy

Total work done is given by: \( U_{1-2} = T_2 - T_1 = \Delta T \)

Modifying this eqn to account for the potential energy terms:

\[
U'_{1-2} + (-\Delta V_g) + (-\Delta V_e) = \Delta T \rightarrow \quad U'_{1-2} = \Delta T + \Delta V
\]

\( U'_{1-2} \) is work of all external forces other than the gravitational and spring forces

\( \Delta T \) is the change in kinetic energy of the particle
\( \Delta V \) is the change in total potential energy

• *More convenient form because only the end point positions of the particle and end point lengths of elastic spring are of significance.*

\[
T_1 + V_1 + U'_{1-2} = T_2 + V_2
\]

If the only forces acting are gravitational, elastic, and *nonworking constraint* forces

\[
T_1 + V_1 = T_2 + V_2 \quad \text{or} \quad E_1 = E_2
\]

\( E = T+V \) is the total mechanical energy of the particle and its attached spring

\( \rightarrow \) This equation expresses the “Law of Conservation of Dynamical Energy”
Kinetics of Particles: Work and Energy

Conservation of Energy

\[ T_1 + V_1 = T_2 + V_2 \quad \text{or} \quad E_1 = E_2 \]

- During the motion, only transformation of KE into PE occurs and it can be vice versa.

A ball of weight \( W \) is dropped from a height \( h \) above the ground (datum).
- PE of the ball is maximum before it is dropped, at which time its KE is zero.

Total mechanical energy of the ball in its initial position is:

\[ E = T_1 + V_1 = 0 + Wh = Wh \]

- When the ball has fallen a distance \( h/2 \), its speed is:

Energy of the ball at mid-height position:

\[ E = T_2 + V_2 = \frac{1}{2} \frac{W}{g} \left( \sqrt{gh} \right)^2 + W \left( \frac{h}{2} \right) = Wh \]

- Just before the ball strikes the ground, its PE=0 and its speed is:

\[ v = \sqrt{2gh} \]

The total mechanical energy of the ball:

\[ E = T_3 + V_3 = \frac{1}{2} \frac{W}{g} \left( \sqrt{2gh} \right)^2 + 0 = Wh \]
Kinetics of Particles: Work and Energy

Potential Energy

Example: A 3 kg slider is released from rest at position 1 and slides with negligible friction in vertical plane along the circular rod. Determine the velocity of the slider as it passes position 2. The spring has an unstretched length of 0.6 m.

Solution:

• Reaction of rod on slider is normal to the motion → does no work → $U'_{1-2} = 0$

Defining the datum to be at the level of position 1

Kinetic Energy: $T_1 = 0$ and $T_2 = \frac{1}{2} (3)(v_2)^2$

Gravitational Potential Energies:

$V_1 = 0$ and $V_2 = -mgh = -3(9.81)(0.6) = -17.66$ J

Initial and final elastic potential energies:

$V_1 = \frac{1}{2}kx_1^2 = 0.5(350)(0.6)^2 = 63$ J and $V_2 = \frac{1}{2}kx_2^2 = 0.5(350)(0.6\sqrt{2} - 0.6)^2 = 10.81$ J

$T_1 + V_1 + U'_{1-2} = T_2 + V_2$

$\rightarrow 0 + (0+63) + 0 = \frac{1}{2} (3)(v_2)^2 + (-17.66 + 10.81)$

$\rightarrow v_2 = 6.82$ m/s
Kinetics of Particles :: Impulse and Momentum

Third approach to solution of Kinetics problems

• *Integrate the equation of motion with respect to time* (rather than disp.)

• Cases where the *applied forces* act for a *very short period of time* (e.g., *Impact loads*) or over *specified intervals of time*

Linear Impulse and Linear Momentum

\[ \Sigma F = m \ddot{v} = \frac{d}{dt} (m \dot{v}) \rightarrow \Sigma F = \dot{G} \]

→ Resultant of all forces acting on a particle equals its time rate of change of linear momentum

Invariability of mass with time!!!
Kinetics of Particles

Linear Impulse and Linear Momentum

Three scalar components of the eqn: 
\[ \Sigma F_x = \dot{G}_x \quad \Sigma F_y = \dot{G}_y \quad \Sigma F_z = \dot{G}_z \]

Linear Impulse-Momentum Principle
• Describes the effect of resultant force on linear momentum of the particle over a finite period of time

Multiplying the eqn by \( dt \) \( \Rightarrow \Sigma F \, dt = dG \) and integrating from \( t_1 \) to \( t_2 \)

\[
\int_{t_1}^{t_2} \Sigma F \, dt = G_2 - G_1 = \Delta G
\]

\( G_1 = \) linear momentum at \( t_1 = mv_1 \)
\( G_2 = \) linear momentum at \( t_1 = mv_2 \)

The product of force and time is defined as Linear Impulse of the Force.

Alternatively:

Initial linear momentum of the body plus the linear impulse applied to it equals its final linear momentum

Impulse integral is a vector!!
Kinetics of Particles

Linear Impulse and Momentum

Impulse-Momentum Equation

\[ G_1 + \int_{t_1}^{t_2} \Sigma F \, dt = G_2 \]

It is necessary to write this eqn in component form and then combine the integrated components:

\[
\begin{align*}
m(v_1)_x + \int_{t_1}^{t_2} \Sigma F_x \, dt &= m(v_2)_x \\
m(v_1)_y + \int_{t_1}^{t_2} \Sigma F_y \, dt &= m(v_2)_y \\
m(v_1)_z + \int_{t_1}^{t_2} \Sigma F_z \, dt &= m(v_2)_z
\end{align*}
\]

The three scalar impulse-momentum eqns are completely independent

Impulse-Momentum Diagram

In the middle drawing linear impulses due to all external forces should be included (except for those forces whose magnitudes are negligible)

Impulse-Momentum diagrams can also show the components
Kinetics of Particles

Linear Impulse and Linear Momentum

*Impulsive Forces*: Large forces of short duration (e.g., hammer impact)

- In some cases, impulsive forces constant over time → they can be brought outside the linear impulse integral.

*Non-impulsive Forces*: can be neglected in comparison with the impulsive forces (e.g., weight of small bodies)

In few cases, graphical or numerical integration is required to be performed.

→ The impulse of this force from $t_1$ to $t_2$
  is the shaded area under the curve

Conservation of Linear Momentum

If resultant force acting on a particle is zero during an interval of time, the impulse momentum equation requires that its linear momentum $\mathbf{G}$ remains constant.

→ The linear momentum of the particle is said to be conserved (in any or all dirn).

$$\Delta \mathbf{G} = 0 \quad \text{or} \quad \mathbf{G}_1 = \mathbf{G}_2$$

This principle is also applicable for motion of two interacting particles with equal and opposite interactive forces.
Example

A particle with a mass of 0.5 kg has a velocity of 10 m/s in the x-direction at time $t = 0$. Forces $F_1$ and $F_2$ act on the particle, and their magnitudes change with time according to the graphical schedule shown. Determine the velocity $v_2$ of the particle at the end of the 3-s interval. The motion occurs in the horizontal $x$-$y$ plane.

Solution:

Construct the impulse-momentum diagram
Example Solution:

Using the impulse-momentum eqns:

\[
\begin{align*}
    m(v_1)_x + \int_{t_1}^{t_2} \Sigma F_x \, dt &= m(v_2)_x \\
    m(v_1)_y + \int_{t_1}^{t_2} \Sigma F_y \, dt &= m(v_2)_y
\end{align*}
\]

\[
\begin{align*}
    0.5(10) - [4(1) + 2(3 - 1)] &= 0.5(v_2)_x \\
    (v_2)_x &= -6 \, \text{m/s} \\
    0.5(0) + [1(2) + 2(3 - 2)] &= 0.5(v_2)_y \\
    (v_2)_y &= 8 \, \text{m/s}
\end{align*}
\]

The velocity \( v_2 \):

\[
\begin{align*}
    v_2 &= -6i + 8j \, \text{m/s} \\
    v_2 &= \sqrt{6^2 + 8^2} = 10 \, \text{m/s}
\end{align*}
\]

\[
\theta_x = \tan^{-1} \frac{8}{-6} = 126.9^\circ
\]
Kinetics of Particles: Linear Impulse and Linear Momentum

Example

The 50-g bullet traveling at 600 m/s strikes the 4-kg block centrally and is embedded within it. If the block slides on a smooth horizontal plane with a velocity of 12 m/s in the direction shown prior to impact, determine the velocity \( \mathbf{v}_2 \) of the block and embedded bullet immediately after impact.

Solution:

The force of impact is internal to the system composed of the block and the bullet. Further, no other external force acts on the system in the plane of the motion.

→ Linear momentum of the system is conserved

→ \( \mathbf{G}_1 = \mathbf{G}_2 \)

\[
0.050(600\mathbf{j}) + 4(12)(\cos 30^\circ \mathbf{i} + \sin 30^\circ \mathbf{j}) = (4 + 0.050)\mathbf{v}_2
\]

\[
\mathbf{v}_2 = 10.26\mathbf{i} + 13.33\mathbf{j} \text{ m/s}
\]

Final velocity and direction:

\[
v_2 = \sqrt{(10.26)^2 + (13.33)^2} = 16.83 \text{ m/s}
\]

\[
\tan \theta = \frac{13.33}{10.26} = 1.299 \quad \theta = 52.4^\circ
\]
Kinetics of Particles

Angular Impulse and Angular Momentum

Velocity of the particle is \( \mathbf{v} = \dot{\mathbf{r}} \)

Momentum of the particle: \( \mathbf{G} = m\mathbf{v} \)

Moment of the linear momentum vector \( m\mathbf{v} \) about the origin \( O \) is defined as Angular Momentum \( \mathbf{H}_O \) of \( P \) about \( O \) and is given by:

\[
\mathbf{H}_O = \mathbf{r} \times m\mathbf{v}
\]

Scalar components of angular momentum:

\[
\mathbf{H}_O = \mathbf{r} \times m\mathbf{v} = m(v_z y - v_y z)\mathbf{i} + m(v_x z - v_z x)\mathbf{j} + m(v_y x - v_x y)\mathbf{k}
\]

\[
H_x = m(v_z y - v_y z)
\]

\[
H_y = m(v_x z - v_z x)
\]

\[
H_z = m(v_y x - v_x y)
\]
Kinetics of Particles

Angular Impulse and Angular Momentum

A 2-D representation of vectors in plane A is shown:

\[ H_O = mvr \sin \theta \]

Magnitude of the moment of \( m\mathbf{v} \) @ \( O \) = linear momentum \( m\mathbf{v} \) times the moment arm

\[ H_O = mvr \sin \theta \]

This is the magnitude of the cross product \( \mathbf{H}_O = \mathbf{r} \times m\mathbf{v} \)

Units of Angular Momentum: \( \text{kg.}(\text{m/s}).\text{m} = \text{kg}.\text{m}^2/\text{s} \) or \( \text{N}.\text{m}.\text{s} \)
Kinetics of Particles

Angular Impulse and Angular Momentum

Rate of Change of Angular Momentum
- To relate moment of forces and angular momentum

Moment of resultant of all forces acting on \( P \) @ origin:
\[
\sum M_O = \mathbf{r} \times \sum \mathbf{F} = \mathbf{r} \times m\mathbf{v}
\]

Differentiating with time:
\[
\dot{\mathbf{H}}_O = \dot{\mathbf{r}} \times m\mathbf{v} + \mathbf{r} \times m\mathbf{\dot{v}} = \mathbf{v} \times m\mathbf{v} + \mathbf{r} \times m\mathbf{\dot{v}}
\]
\( \mathbf{v} \) and \( m\mathbf{v} \) are parallel vectors \( \rightarrow \mathbf{v} \times m\mathbf{v} = 0 \)

\( \rightarrow \sum M_O = \dot{\mathbf{H}}_O \) Using this vector equation, moment of forces and angular momentum are related

\( \rightarrow \) Moment of all forces @ \( O \) = time rate of change of angular momentum

Scalar components:
\[
\sum M_{Ox} = \dot{H}_{Ox}, \quad \sum M_{Oy} = \dot{H}_{Oy}, \quad \sum M_{Oz} = \dot{H}_{Oz}
\]
Angular Impulse-Momentum Principle

This eqn gives the instantaneous relation between moment and time rate of change of angular momentum

\[ \Sigma M_O \, dt = dH_O \]

Integrating:

\[ \int_{t_1}^{t_2} \Sigma M_O \, dt = (H_O)_2 - (H_O)_1 = \Delta H_O \]

The product of moment and time is defined as the angular impulse.

The total angular impulse on \( m \) about the fixed point \( O \) equals the corresponding change in the angular momentum of \( m \) about \( O \).

Alternatively:

Initial angular momentum of the particle plus the angular impulse applied to it equals the final angular momentum.
Angular Impulse and Angular Momentum

Angular Impulse-Momentum Principle
In the component form:
The \( x \)-component of this eqn:
Similarly other components can be written

Plane Motion Applications
- In most applications, plane motions are encountered instead of 3-D motion.
- Simplifying the eqns
Using the scalar form of the principle betn 1 and 2:

\[
(H_O)_1 + \int_{t_1}^{t_2} \Sigma M_O \, dt = (H_O)_2
\]
\[
mv_1d_1 + \int_{t_1}^{t_2} \Sigma Fr \sin \theta \, dt = mv_2d_2
\]
Kinetics of Particles

Angular Impulse and Angular Momentum

Conservation of Angular Momentum
If the resulting moment @ a fixed point $O$ of all forces acting on a particle is zero during an interval of time, the angular momentum of the particle about that point remain constant.

$\Delta H_O = 0 \quad \text{or} \quad (H_O)_1 = (H_O)_2$

→ Principle of Conservation of Angular Momentum
→ Also valid for motion of two interacting particles with equal and opposite interacting forces
Kinetics of Particles

Angular Impulse and Angular Momentum

Example

A small sphere has the position and velocity indicated in the figure and is acted upon by the force $F$. Determine the angular momentum $H_O$ about point $O$ and the time derivative $\dot{H}_O$.

Solution

\[
H_O = r \times m v \\
= (3i + 6j + 4k) \times 2(5j) \\
= -40i + 30k \text{ N} \cdot \text{m/s}
\]

\[
\dot{H}_O = M_O \\
= r \times F \\
= (3i + 6j + 4k) \times 10k \\
= 60i - 30j \text{ N} \cdot \text{m}
\]
Kinetics of Particles

Impact

• Collision between two bodies during a very short period of time.
• Generation of large contact forces (impulsive) acting over a very short interval of time.
• Complex phenomenon (material deformation and recovery, generation of heat and sound)
• Line of Impact is the common normal to the surfaces in contact during impact.

Impact primarily classified as two types:

- **Central Impact**: Mass centers of two colliding bodies are located on line of impact → Current chapter deals with central impact of two particles.
- **Eccentric Impact**: Mass centers are not located on line of impact.

![Central Impact](image1)

![Eccentric Impact](image2)
Kinetics of Particles

Central Impact
- Can be classified into two types
  - **Direct Central Impact (or Direct Impact)**
    - Velocities of the two particles are directed along the line of impact
    - Direction of motion of the particles will also be along the line of impact
  - **Oblique Central Impact (or Oblique Impact)**
    - Velocity and motion of one or both particles is at an angle with the line of impact.
    - Initial and final velocities are not parallel.
Kinetics of Particles: Impact

Direct Central Impact
Collinear motion of two spheres \((v_1 > v_2)\)
- Collision occurs with contact forces directed along the line of impact (line of centers)
- Deformation of spheres increases until contact area ceases to increase. Both spheres move with the same velocity.
- Period of restoration during which the contact area decreases to zero
- After the impact, spheres will have different velocities \((v'_1 \text{ & } v'_2)\) with \(v'_1 < v'_2\)

During impact, contact forces are equal and opposite. Further, there are no impulsive external forces \(\rightarrow\) linear momentum of the system remains unchanged.

Applying the law of conservation of linear momentum:

\[
m_1v_1 + m_2v_2 = m_1v'_1 + m_2v'_2
\]

Assumptions
- Particles are perfectly smooth and frictionless.
- Impulses created by all forces (other than the internal forces of contact) are negligible compared to the impulse created by the internal impact force.
- No appreciable change in position of mass centers during the impact.
Kinetics of Particles: Impact

Coefficient of Restitution

The momentum eqn contains 2 unknowns \( v'_1 \) & \( v'_2 \) (assuming that \( v_1 \) & \( v_2 \) are known)

\[
m_1 v_1 + m_2 v_2 = m_1 v'_1 + m_2 v'_2
\]

\( \rightarrow \) Another equation is required

\( \rightarrow \) Coefficient of Restitution (\( e \))

\[
e = \frac{\text{magnitude of the Restoration Impulse}}{\text{magnitude of the Deformation Impulse}}
\]

Using the definition of impulse:

\[
e = \frac{\int_{t_0}^{t} F_r \, dt}{\int_{0}^{t_0} F_d \, dt}
\]

\( F_r \) and \( F_d \) = magnitudes of the contact forces during the restoration and deformation periods.

\( t_0 \) = time for the deformation

\( t \) = total time of contact

Further using Linear Impulse-Momentum Principle, we can write this equation in terms of change in momentum

\[
\int_{t_1}^{t_2} \Sigma F \, dt = G_2 - G_1 = \Delta G
\]
Kinetics of Particles: Impact

Coefficient of Restitution

For Particle 1:

\[ e = \frac{\int_{t_0}^{t} F_r \, dt}{\int_{t_0}^{t} F_d \, dt} = \frac{m_1[-v_1' - (-v_0)]}{m_1[-v_0 - (-v_1)]} = \frac{v_0 - v_1'}{v_1 - v_0} \]

For Particle 2:

\[ e = \frac{\int_{t_0}^{t} F_r \, dt}{\int_{t_0}^{t} F_d \, dt} = \frac{m_2(v_2' - v_0)}{m_2(v_0 - v_2)} = \frac{v_2' - v_0}{v_0 - v_2} \]

Eliminating \( v_0 \) between the two expressions:

\[ e = \frac{v_2' - v_1'}{v_1 - v_2} = \frac{|\text{relative velocity of separation}|}{|\text{relative velocity of approach}|} \]

If \( v_1, v_2, \) and \( e \) are known, the final velocities \( v'_1 \) & \( v'_2 \) can be obtained using the two eqns \( \rightarrow \) eqn for \( e \) and momentum eqn:

\[ m_1v_1 + m_2v_2 = m_1v'_1 + m_2v'_2 \]
Kinetics of Particles: Impact

Coefficient of Restitution

Energy Loss during impact
- Impact always associated with energy loss (heat, inelastic deformation, etc)
- Energy loss may be determined by finding the change in the KE of the system before and after the impact.

Classical theory of impact:
\[ e = 1 \rightarrow \text{Elastic Impact (no energy loss)} \]
- Put \( e = 1 \) in the eqn \( v'_2 - v'_1 = v_1 - v_2 \)
- Relative velocities before & after impact are equal
- Particles move away after impact with the same velocity with which they approached each other before impact.

\[ e = 0 \rightarrow \text{Inelastic or Plastic Impact (max energy loss)} \]
- Put \( e = 0 \) in the eqn \( v'_1 = v'_2 \)
- The particles stick together after collision and move with a common velocity

Real conditions lie somewhere between these extremes
- \( e \) varies with impact velocity, and size and shape of the colliding bodies.
- \( e \) is considered constant for given geometries and a given combination of contacting materials.
- \( e \) approaches unity as the impact velocity approaches zero.
Kinetics of Particles: Impact

Oblique Impact

• In-plane initial and final velocities are not parallel
• Choosing the $n$-axis along the line of impact, and the $t$-axis along the common tangent.
• Directions of velocity vectors measured from $t$-axis.

Initial velocity components are:

$$(v_1)_n = -v_1 \sin \theta_1, \quad (v_1)_t = v_1 \cos \theta_1$$
$$(v_2)_n = v_2 \sin \theta_2, \quad (v_2)_t = v_2 \cos \theta_2$$

→ Four unknowns $(v'_1)_n$, $(v'_1)_t$, $(v'_2)_n$, $(v'_2)_t$

→ Four equations are required.

Particles are assumed to be perfectly smooth and frictionless → the only impulses exerted on the particles during the impact are due to the internal forces directed along the line of impact ($n$-axis).

Impact forces acting on the two particles are equal and opposite: $\mathbf{F}$ and $-\mathbf{F}$ (variation during impact is shown).

→ Since there are no impulsive external forces, total momentum of the system (both particles) is conserved along the $n$-axis.
Kinetics of Particles: Impact

Oblique Impact:

• total momentum of the system is conserved along the $n$-axis

$$m_1(v_1)_n + m_2(v_2)_n = m_1(v_1')_n + m_2(v_2')_n \quad (1)$$

• Component along $t$-axis of the momentum of each particle, considered separately, is conserved (since there is no impulse on particles along $t$-direction)

$$m_1(v_1)_t = m_1(v_1')_t$$
$$m_2(v_2)_t = m_2(v_2')_t \quad (2 \text{ and } 3)$$

→ $t$-component of velocity of each particle remains unchanged.

• Coefficient of the Restitution is the positive ratio of the recovery impulse to the deformation impulse → the eqn will be applied to the velocity components along $n$-direction.

$$e = \frac{(v_2')_n - (v_1')_n}{(v_1)_n - (v_2)_n} \quad (4)$$

Once the four final velocities are determined, angles $\theta_1'$ and $\theta_2'$ can be easily determined.
Example:
The ram of a pile driver has a mass of 800 kg and is released from rest 2 m above the top of the 2400-kg pile. If the ram rebounds to a height of 0.1 m after impact with the pile, calculate (a) the velocity $v_p'$ of the pile immediately after impact, (b) the coefficient of restitution $e$, and (c) the percentage loss of energy due to the impact.

Solution:
Direct Central Impact
Kinetics of Particles: Impact

Example:
Energy is conserved during free fall: \( T_1 + V_1 = T_2 + V_2 \) or \( E_1 = E_2 \)
Initial and final velocities of the ram can be calculated from:
\[
v^2 = v_0^2 + 2a_c(y - y_0) \rightarrow v = \sqrt{2gh}
\]

\[v_r = \sqrt{2(9.81)(2)} = 6.26 \text{ m/s} \quad v'_r = \sqrt{2(9.81)(0.1)} = 1.401 \text{ m/s}\]

Conservation of the momentum of the system of ram and pile:

\[G_1 = G_2 \]

\[800(6.26) + 0 = 800(-1.401) + 2400v'_p \]

\[\rightarrow v'_p = 2.55 \text{ m/s} \quad [\text{impulses of the weights are neglected}]\]

Coefficient of Restitution

\[e = \frac{|\text{rel. vel. separation}|}{|\text{rel. vel. approach}|}\]

\[e = \frac{2.55 + 1.401}{6.26 + 0} = 0.631\]

Loss of energy due to impact can be calculated from difference in KE of system
KE just before impact: \[T = \frac{1}{2} (800)(6.26)^2 = 15675 \text{ J}\]
Since the energy is conserved this can also be calculated from \( PE=mg_h \)
KE just after impact: \[T' = \frac{1}{2} (800)(1.401)^2 + \frac{1}{2} (2400)(2.55)^2 = 8588 \text{ J}\]
% loss of energy: \[((15675-8588)/15675)x(100) = 45.2\%\]