Kinematics of Particles

Plane Curvilinear Motion
Motion of a particle along a curved path which lies in a single plane.

For a short time during take-off and landing, planes generally follow plane curvilinear motion.
Kinematics of Particles

Plane Curvilinear Motion:

Between $A$ and $A'$:
Average velocity of the particle: $\mathbf{v}_{av} = \frac{\Delta \mathbf{r}}{\Delta t}$
→ A vector whose direction is that of $\Delta \mathbf{r}$ and whose magnitude is magnitude of $\Delta \mathbf{r}/\Delta t$

Average speed of the particle = $\frac{\Delta s}{\Delta t}$

*Instantaneous velocity of the particle is defined as the limiting value of the average velocity as the time interval approaches zero* →

$\mathbf{v} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t}$

→ $\mathbf{v}$ is always a vector tangent to the path

Extending the definition of derivative of a scalar to include vector quantity:

$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}}$

Derivative of a vector is a vector having a magnitude and a direction.

Magnitude of $\mathbf{v}$ is equal to speed (scalar)

$v = |\mathbf{v}| = \frac{ds}{dt} = \dot{s}$
**Kinematics of Particles**

**Plane Curvilinear Motion**

Magnitude of the derivative:

\[
\left| \frac{d\mathbf{r}}{dt} \right| = |\mathbf{r}| = \mathbf{s} = |\mathbf{v}| = \mathbf{v}
\]

→ Magnitude of the velocity or the speed

Derivative of the magnitude:

\[
\frac{d|\mathbf{r}|}{dt} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}}
\]

→ Rate at which the length of the position vector is changing

Velocity of the particle at \( A \) → tangent vector \( \mathbf{v} \)
Velocity of the particle at \( A' \) → tangent vector \( \mathbf{v}' \)

→ \( \mathbf{v}' - \mathbf{v} = \Delta \mathbf{v} \)

→ \( \Delta \mathbf{v} \) Depends on both the change in magnitude of \( \mathbf{v} \) and on the change in direction of \( \mathbf{v} \).
Kinematics of Particles

Plane Curvilinear Motion

Between A and A’:
Average acceleration of the particle: \( a_{av} = \frac{\Delta v}{\Delta t} \)
\( \rightarrow \) A vector whose direction is that of \( \Delta v \) and whose magnitude is the magnitude of \( \Delta v/\Delta t \)

Instantaneous accln of the particle is defined as the limiting value of the average accln as the time interval approaches zero →

By definition of the derivative:

\( \vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} \)

→ In general, direction of the acceleration of a particle in curvilinear motion neither tangent to the path nor normal to the path.
→ Acceleration component normal to the path points toward the center of curvature of the path.
Acceleration has the same relation to velocity as the velocity has to the position vector.
Kinematics of Particles

Plane Curvilinear Motion

Derivatives and Integration of Vectors:
same rules as for scalars

\[
\frac{d \mathbf{P}}{dt} = \dot{\mathbf{P}} = \dot{P}_x \mathbf{i} + \dot{P}_y \mathbf{j} + \dot{P}_z \mathbf{k}
\]

\[
\frac{d(\mathbf{P} \cdot \mathbf{u})}{dt} = \mathbf{P} \ddot{\mathbf{u}} + \dot{\mathbf{P}} \cdot \mathbf{u}
\]

\[
\frac{d(\mathbf{P} \cdot \mathbf{Q})}{dt} = \mathbf{P} \cdot \dot{\mathbf{Q}} + \dot{\mathbf{P}} \cdot \mathbf{Q}
\]

\[
\frac{d(\mathbf{P} \times \mathbf{Q})}{dt} = \mathbf{P} \times \dot{\mathbf{Q}} + \dot{\mathbf{P}} \times \mathbf{Q}
\]

\[d\tau = dx
dy
dz\]

\[
\int \mathbf{V} d\tau = \mathbf{i} \int V_x d\tau + \mathbf{j} \int V_y d\tau + \mathbf{k} \int V_z d\tau
\]

\(\mathbf{V}\) is a function of \(x, y, \) and \(z,\) and an element of volume is \(d\tau = dx\ dy\ dz\) Integral of \(\mathbf{V}\) over the volume is equal to the vector sum of the three integrals of its components.
Kinematics of Particles

Plane Curvilinear Motion

Three coordinate systems are commonly used for describing the vector relationships (for plane curvilinear motion of a particle):

1. Rectangular Coordinates $\rightarrow x-y$
2. Normal and tangential coordinates $\rightarrow n-t$
3. Polar coordinates $\rightarrow r-\theta$ (special case of 3-D motion in which cylindrical coordinates $r, \theta, z$ are used)

Choice of coordinate systems depends on
$\rightarrow$ the manner in which the motion is generated
$\rightarrow$ or the form in which the data is specified.
Kinematics of Particles: Plane Curvilinear Motion

Rectangular Coordinates (x-y)

If all motion components are directly expressible in terms of horizontal and vertical coordinates

\[
\mathbf{r} = x\mathbf{i} + y\mathbf{j} \\
\mathbf{v} = \dot{\mathbf{r}} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} \\
\mathbf{a} = \ddot{\mathbf{r}} = \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j}
\]

\[v_x = \dot{x}, \quad v_y = \dot{y} \quad \text{and} \quad a_x = \ddot{x}, \quad a_y = \ddot{y}\]

\[v^2 = v_x^2 + v_y^2 \quad \mathbf{v} = \sqrt{v_x^2 + v_y^2} \quad \tan \theta = \frac{v_y}{v_x}
\]

\[a^2 = a_x^2 + a_y^2 \quad \mathbf{a} = \sqrt{a_x^2 + a_y^2}
\]

Also, \(\frac{dy}{dx} = \tan \theta = \frac{v_y}{v_x}\)

Time derivatives of the unit vectors are zero because their magnitude and direction remains constant.
Kinematics of Particles: **Plane Curvilinear Motion**

**Rectangular Coordinates (x-y)**

**Projectile Motion** → An important application

**Assumptions:** neglecting aerodynamic drag, Neglecting curvature and rotation of the earth, and altitude change is small enough such that $g$ can be considered to be constant → Rectangular coordinates are useful for the trajectory analysis

For the axes shown in the figure, the acceleration components are: $a_x = 0$, $a_y = -g$

Integrating these eqns for the condition of constant accln (slide 11) will give us equations necessary to solve the problem.
Kinematics of Particles: Plane Curvilinear Motion

Rectangular Coordinates (x-y)

Projectile Motion

**Horizontal Motion:** \( a_x = 0 \)

Integrating this eqn for constant accln condition

\[
\begin{align*}
\vec{v} &= \vec{v}_0 + \vec{a}t \\
x &= x_0 + v_0 t + \frac{1}{2} \vec{a} t^2 \\
v^2 &= v_0^2 + 2a(x-x_0) \\
\end{align*}
\]

**Vertical Motion:** \( a_y = -g \)

Integrating this eqn for constant accln condition

\[
\begin{align*}
\vec{v} &= \vec{v}_0 + \vec{a}t \\
y &= y_0 + v_0 t + \frac{1}{2} \vec{a} t^2 \\
v^2 &= v_0^2 + 2a(y-y_0) \\
\end{align*}
\]

Subscript zero denotes initial conditions: \( x_0 = y_0 = 0 \)

For the conditions under discussion:
- \( x \)- and \( y \)- motions are independent
- Path is parabolic
Kinematics of Particles: Plane Curvilinear Motion

Normal and Tangential Coordinates \((n-t)\)

Common descriptions of curvilinear motion uses Path Variables: measurements made along the tangent and normal to the path of the particle.

- **Positive \(n\) direction**: towards the center of curvature of the path

**Velocity and Acceleration**

\[ \mathbf{e}_n = \text{unit vector in the } n\text{-direction at point } A \]

\[ \mathbf{e}_t = \text{unit vector in the } t\text{-direction at point } A \]

During differential increment of time \(dt\), the particle moves a differential distance \(ds\) from \(A\) to \(A'\).

\[ \rho = \text{radius of curvature of the path at } A' \]

\[ ds = \rho \, d\beta \]

**Magnitude of the velocity**: \(v = ds/dt = \rho \, d\beta/dt\)

- **In vector form**

\[ \mathbf{v} = v \mathbf{e}_t = \rho \frac{d\beta}{dt} \mathbf{e}_t \]

Differentiating:

\[ \mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d(v \mathbf{e}_t)}{dt} = v \frac{d\mathbf{e}_t}{dt} + \frac{dv}{dt} \mathbf{e}_t \]

Unit vector \(\mathbf{e}_t\) has non-zero derivative because its direction changes.
Kinematics of Particles: Plane Curvilinear Motion

Normal and Tangential Coordinates \((n-t)\)

Determination of \(\dot{e}_t;\)
- change in \(e_t\) during motion from \(A\) to \(A'\)
- The unit vector changes to \(e'_t\)
The vector difference \(de_t\) is shown in the bottom figure.
  - In the limit \(de_t\) has magnitude equal to length of the arc \(|e_t|\) \(d\beta = d\beta\)
  - Direction of \(de_t\) is given by \(e_n\)
- We can write: \(de_t = e_n d\beta\)

Dividing by \(dt\): \(de_t/\dt = e_n (d\beta/\dt) e_n \) \(\Rightarrow \dot{e}_t = \beta e_n\)

Substituting this and \(v = \rho (d\beta/\dt) = \frac{v}{\rho} \) in equation for acceleration:

\[
\begin{align*}
a &= \frac{dv}{dt} = \frac{d(v e_t)}{dt} = v \dot{e}_t + \dot{v} e_t \\
a &= \frac{v^2}{\rho} e_n + \dot{v} e_t \\
\end{align*}
\]

Here:

\[
\begin{align*}
a_n &= \frac{v^2}{\rho} = \rho \beta^2 = v \dot{\beta} \\
a_t &= \dot{v} = \ddot{s} \\
a &= \sqrt{a_n^2 + a_t^2}
\end{align*}
\]
Kinematics of Particles: Plane Curvilinear Motion

Normal and Tangential Coordinates (n-t)

Important Equations

\[ v = \rho \beta \]

\[ a = \frac{v^2}{\rho} e_n + \dot{\rho} e_t \]

- In \( n-t \) coordinate system, there is no component of velocity in the normal direction because of constant \( \rho \) for any section of curve (normal velocity would be rate of change of \( \rho \)).

- Normal component of the acceleration \( a_n \) is always directed towards the center of the curvature \( \rightarrow \) sometimes referred as centripetal acceleration.
  - If the particle moves with constant speed, \( a_t = 0 \), and \( a = a_n = \frac{v^2}{\rho} \)

  \( \rightarrow a_n \) represents the time rate of change in the dirn of vel.

- Tangential component \( a_t \) will be in the +ve \( t \)-dirn of motion if the speed \( v \) is increasing, and in the -ve \( t \)-direction if the speed is decreasing.
  - If the particle moves in a straight line, \( \rho = \infty \)
  
  \( a_n = 0 \), and \( a = a_t = \dot{v} = \ddot{s} \)

  \( \rightarrow a_t \) represents the time rate of change in the magnitude of velocity.

Directions of tangential components of acceleration are shown in the figure.
Kinematics of Particles: Plane Curvilinear Motion

Normal and Tangential Coordinates ($n\text{-}t$)

Circular Motion: Important special case of plane curvilinear motion
- Radius of curvature becomes constant (radius $r$ of the circle).
- Angle $\beta$ is replaced by the angle $\theta$ measured from any radial reference to $OP$.

Velocity and acceleration components for the circular motion of the particle:

\[
\begin{align*}
v &= \rho \dot{\beta} \\
a_n &= \frac{v^2}{\rho} = \rho \ddot{\beta} = v \dot{\beta} \\
a_t &= \dot{v} = \ddot{s} \\
a &= \sqrt{a_n^2 + a_t^2}
\end{align*}
\]

General motion

Circular motion
Kinematics of Particles: Plane Curvilinear Motion

Rectangular Coordinates (x-y)

Example
The curvilinear motion of a particle is defined by \( v_x = 50 - 16t \) and \( y = 100 - 4t^2 \). At \( t = 0 \), \( x = 0 \). \( v_x \) is in m/s\(^2\), \( x \) and \( y \) are in m, and \( t \) is in s. Plot the path of the particle and determine its velocity and acceleration at \( y = 0 \).

Solution:

\[
\int dx = \int v_x \, dt \\
\int_0^x dx = \int_0^t (50 - 16t) \, dt \\
x = 50t - 8t^2 \text{ m}
\]

\[
[a_x = \dot{v}_x] \\
a_x = \frac{d}{dt} (50 - 16t) \\
a_x = -16 \text{ m/s}^2
\]

\[
[v_y = \dot{y}] \\
v_y = \frac{d}{dt} (100 - 4t^2) \\
v_y = -8t \text{ m/s}
\]

\[
[a_y = \dot{v}_y] \\
a_y = \frac{d}{dt} (-8t) \\
a_y = -8 \text{ m/s}^2
\]

Calculate \( x \) and \( y \) for various \( t \) values and plot.

ME101 - Division III
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Kinematics of Particles: Plane Curvilinear Motion

Rectangular Coordinates (x-y)

Example Solution:

When \( y = 0 \), \( 0 = 100 - 4t^2 \) \( \Rightarrow t = 5 \) s

\[
\begin{align*}
v_x &= 50 - 16(5) = -30 \text{ m/s} \\
v_y &= -8(5) = -40 \text{ m/s} \\
v &= \sqrt{(-30)^2 + (-40)^2} = 50 \text{ m/s} \\
a &= \sqrt{(-16)^2 + (-8)^2} = 17.89 \text{ m/s}^2
\end{align*}
\]

\[
\begin{align*}
v &= -30\mathbf{i} - 40\mathbf{j} \text{ m/s} \\
a &= -16\mathbf{i} - 8\mathbf{j} \text{ m/s}^2
\end{align*}
\]
Kinematics of Particles: Plane Curvilinear Motion

Rectangular Coordinates (x-y)

Example: The rider jumps off the slope at $30^0$ from a height of 1 m, and remained in air for 1.5 s. Neglect the size of the bike and of the rider. Determine:

(a) the speed at which he was travelling off the slope,
(b) the horizontal distance he travelled before striking the ground, and
(c) the maximum height he attains.

Solution:
Let the origin of the coordinates be at A.
Rectangular Coordinates (x-y)

Example:
Solution: For projectile motion:
\( a_x = 0, \ a_y = -g = -9.81 \text{ m/s}^2 \rightarrow \text{Constant Acceleration} \)

(a) speed at which he was travelling off the slope?

Let \( v_0 \) be the initial velocity of the bike at A.
For vertical Motion: \( a_y = -g; \text{ subsequent integrations will give following equations} \)

\[
\begin{align*}
v &= v_0 + at \\
y &= y_0 + v_0 t + \frac{1}{2}at^2 \\
v^2 &= v_0^2 + 2a(y - y_0)
\end{align*}
\]

Using second eqn: \(-1 = 0 + (v_0)_y(1.5) - 0.5(9.81)(1.5)^2 \)
Initial velocity along y-direction \((v_0)_y = v_0 \sin 30 = 0.5v_0 \)
\rightarrow \(-1 = 0 + 0.5v_0(1.5) - 0.5(9.81)(1.5)^2 \)
\rightarrow Initial Velocity of the bike: \( v_0 = 13.38 \text{ m/s} \) (velocity at A)
Kinematics of Particles: Plane Curvilinear Motion

Rectangular Coordinates (x-y)

Example:
Solution: For projectile motion:
\( a_x = 0, \ a_y = -g = -9.81 \text{ m/s}^2 \rightarrow \text{Constant Acceleration} \)

(b) horizontal distance he travelled before striking the ground?

Let \( R \) be the horizontal distance between \( A \) and \( B \).
For horizontal Motion: \( a_x = 0; \) subsequent integrations will give following equations

\[
\begin{align*}
v &= v_0 + at \quad \Rightarrow \quad v_x = (v_0)_x \\
x &= x_0 + v_0 t + \frac{1}{2} at^2 \quad \Rightarrow \quad x = x_0 + (v_0)_x t \\
v^2 &= v_0^2 + 2a(x - x_0) \quad \Rightarrow \quad v_x = (v_0)_x
\end{align*}
\]

Using second eqn: \( R = 0 + (v_0)_x(1.5) = 13.38 \cos 30(1.5) \)
\( \rightarrow \) Horz distance: \( R = 17.4 \text{ m} \)
Example:
Solution: For projectile motion:
\[ a_x = 0, \ a_y = -g = -9.81 \text{ m/s}^2 \rightarrow \text{Constant Acceleration} \]

(c) Maximum height attained by the bike?

Let \((h - 1) \text{ m}\) be the maximum height attained from \(x\)-axis at point \(C\).

For Vertical Motion: \(a_y = -g\),

\[ v = v_0 + at \quad \Rightarrow \quad v_y = (v_0)_y - gt \]
\[ y = y_0 + v_0 t + \frac{1}{2} at^2 \quad \Rightarrow \quad y = y_0 + (v_0)_y t - \frac{1}{2} gt^2 \]
\[ v^2 = v_0^2 + 2a(y - y_0) \quad \Rightarrow \quad v_y^2 = (v_0)_y^2 - 2g(y - y_0) \]

Using the third eqn between \(A\) and \(C\): All the quantities are known except the height of point \(C\) \((y = h-1)\) and the velocity at point \(C\) \(\rightarrow v_y = 0\) at \(C\)

\[ 0 = (0.5 \times 13.38)^2 - 2(9.81)(h - 1 - 0) \]
\[ \rightarrow h = 3.28 \text{ m} \text{ (total height attained above ground level)} \]
Normal and Tangential Coordinates (n-t)

Example: At the position shown, the driver applies brakes to produce a uniform deceleration. Speed of the car is 100 km/h at A (bottom of the dip), and 50 km/h at C (top of the hump). Distance between A and C is 120 m along the road. Passengers experience a total acceleration of 3 m/s² at A. Radius of curvature of the hump at C is 150 m. Calculate:
(a) radius of curvature at A
(b) total acceleration at inflection point B, and
(c) total acceleration at C.
Kinematics of Particles: Plane Curvilinear Motion

Normal and Tangential Coordinates (n-t)

Example Solution:
Converting the units of Velocity:
\(v_A = 100 \text{ km/h} \left[ \frac{1000}{(60 \times 60)} \right] = 27.8 \text{ m/s}\)
\(v_C = 50 \text{ km/h} \left[ \frac{1000}{(60 \times 60)} \right] = 13.89 \text{ m/s}\)

For Constant Deceleration, we can use the following formulae:

\[v = v_0 + at\]
\[x = x_0 + v_0 t + \frac{1}{2}at^2\]
\[v^2 = v_0^2 + 2a(x-x_0)\]

Using the third equation between \(A\) and \(C\) to find the constant deceleration of the car:
\[(13.89)^2 = (27.8)^2 + 2a(120 - 0)\]
\(\rightarrow a = -2.41 \text{ m/s}^2\)
This acceleration is the tangential component of the total acceleration \(\rightarrow a_t = -2.41 \text{ m/s}^2\)

(a) radius of curvature at \(A\)?
Total accln at \(A\) is given as: \(a = 3 \text{ m/s}^2\)
Using the third eqn: \((3)^2 = (a_n)^2 + (-2.41)^2\)
\(\rightarrow a_n = 1.785 \text{ m/s}^2\)
Using the first eqn: \(\rho_A = (27.8)^2/1.785 \rightarrow \rho_A = 432 \text{ m}\)
Kinematics of Particles: Plane Curvilinear Motion

Normal and Tangential Coordinates ($n$-$t$)

Example

Solution:

$v_A = 100 \text{ km/h} \left[ \frac{1000}{(60 \times 60)} \right] = 27.8 \text{ m/s}

v_C = 50 \text{ km/h} \left[ \frac{1000}{(60 \times 60)} \right] = 13.89 \text{ m/s}

(b) total acceleration at inflection point $B$?

Tangential component of acceleration at $B$, $a_t = -2.41 \text{ m/s}^2$

At inflection point radius of curvature is infinity, Therefore, normal component of acceleration, $a_n = 0$

→ Total acceleration at $B$: $a = a_t = -2.41 \text{ m/s}^2$

(c) total acceleration at $C$?

Tangential component of acceleration at $C$, $a_t = -2.41 \text{ m/s}^2$

Normal component can be found from first eqn:

$a_n = \frac{v^2}{\rho} = \frac{13.89^2}{150} = 1.286 \text{ m/s}^2$

Total acceleration at $C$: $a^2 = (1.286)^2 + (-2.41)^2$

→ Total acceleration at $C$: $a = 2.73 \text{ m/s}^2$