Virtual Work

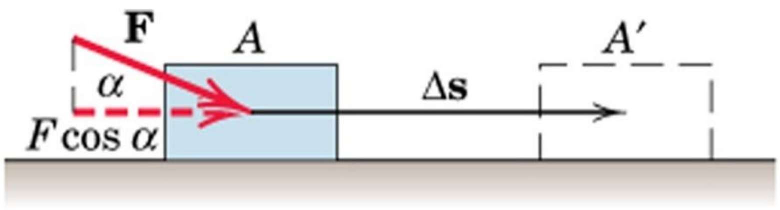
Method of Virtual Work

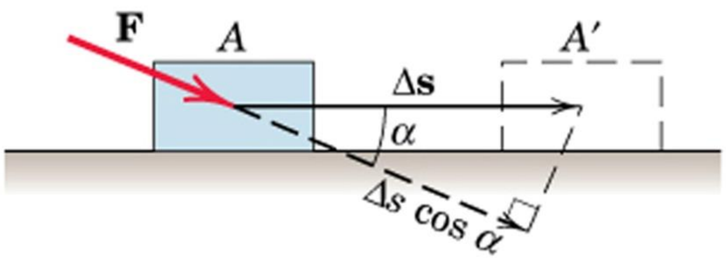
- Previous methods (FBD, $\sum F$, $\sum M$) are generally employed for a body whose equilibrium position is known or specified
- For problems in which bodies are composed of interconnected members that can move relative to each other,
- → various equilibrium configurations are possible and must be examined.
- → previous methods can still be used but are not the direct and convenient.
- **Method of Virtual Work** is suitable for analysis of multi-link structures (pin-jointed members) which change configuration
- **effective** when a **simple relation** can be found among the **disp.** of the **pts of application** of **various forces** involved
- → based on the **concept of work** done by a **force**
- → enables us to examine **stability of systems** in **equilibrium**

Scissor Lift Platform

Virtual Work

Work done by a Force (*U*)

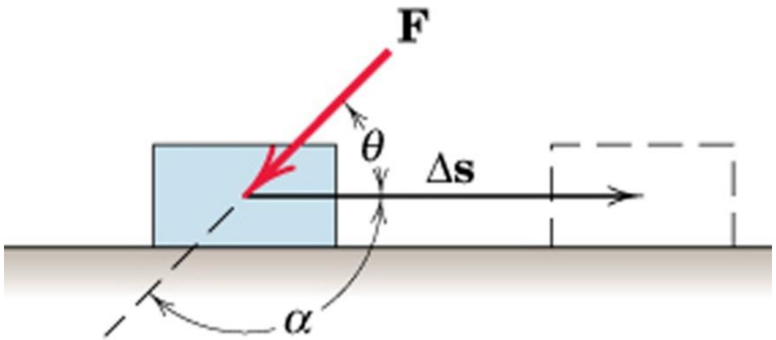




U = work done by the component of the force in the direction of the displacement times the displacement

$U = (F \cos \alpha)\, \Delta s$ or $U = F(\Delta s \cos \alpha)$

Since same results are obtained irrespective of the direction in which we resolve the vectors → Work is a scalar quantity



+*U* → Force and Disp in same direction

- *U* → Force and Disp in opposite direction

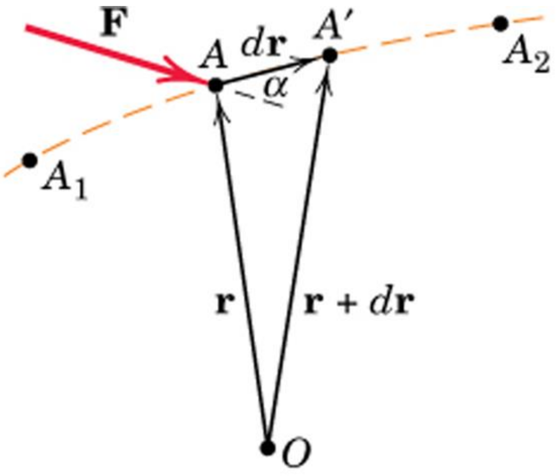
$$U = (F \cos \alpha)\, \Delta s = -(F \cos \theta)\, \Delta s$$

Virtual Work

Generalized Definition of Work

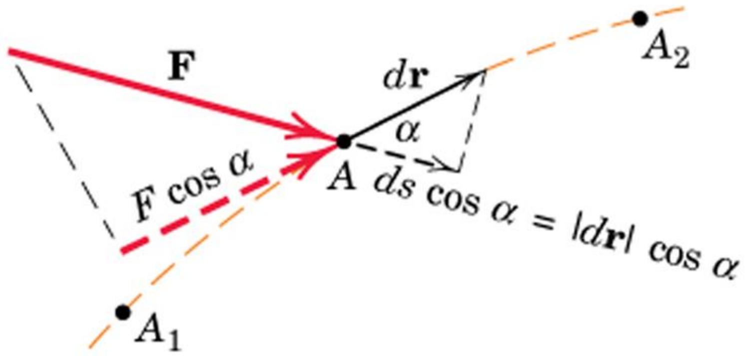
Work done by **F** during displacement $d\mathbf{r}$

$$dU = \mathbf{F} \cdot d\mathbf{r}$$



→ $$dU = F\, ds \cos \alpha$$

Expressing **F** and $d\mathbf{r}$ in terms of their rectangular components

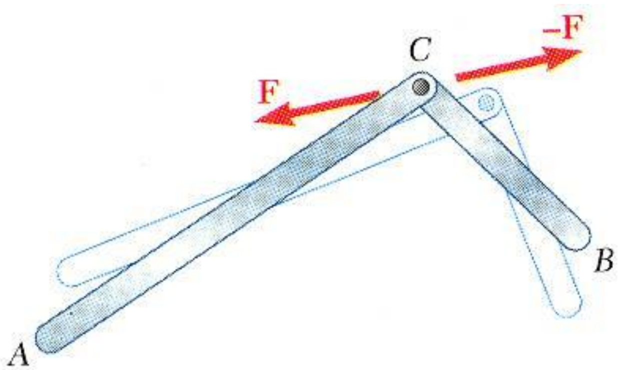


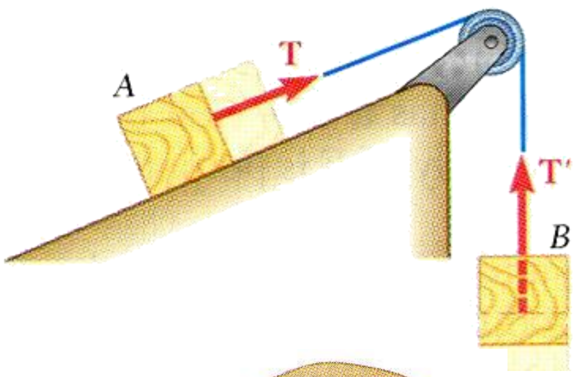
$$dU = (\mathbf{i}F_x + \mathbf{j}F_y + \mathbf{k}F_z) \cdot (\mathbf{i}\, dx + \mathbf{j}\, dy + \mathbf{k}\, dz)$$
$$= F_x\, dx + F_y\, dy + F_z\, dz$$

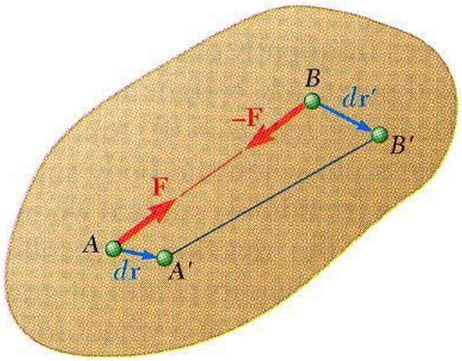
Total work done by **F** from A_1 to A_2

$$U = \int \mathbf{F} \cdot d\mathbf{r} = \int (F_x\, dx + F_y\, dy + F_z\, dz)$$ → $$U = \int F \cos \alpha\, ds$$

Virtual Work: Work done by a Force







Forces which do no work:

- forces applied to fixed points ($ds = 0$)
- forces acting in a dirn normal to the disp ($\cos\alpha = 0$)
- reaction at a frictionless pin due to rotation of a body around the pin
- reaction at a frictionless surface due to motion of a body along the surface
- weight of a body with cg moving horizontally
- friction force on a wheel moving without slipping

Sum of work done by several forces may be zero:

- bodies connected by a frictionless pin
- bodies connected by an inextensible cord
- internal forces holding together parts of a rigid body

Virtual Work

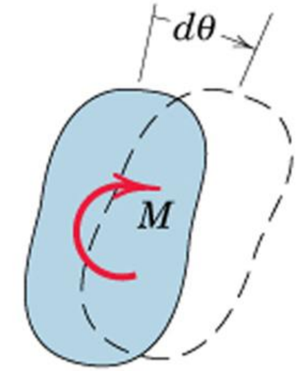
Work done by a Couple (U)

Small rotation of a rigid body:

- translation to *A'B'*

  → work done by F during disp AA' will be equal and opposite to work done by $-F$ during disp BB'

  → total work done is zero



- rotation of *A'* about *B'* to *A"*

→work done by F during disp AA'' :

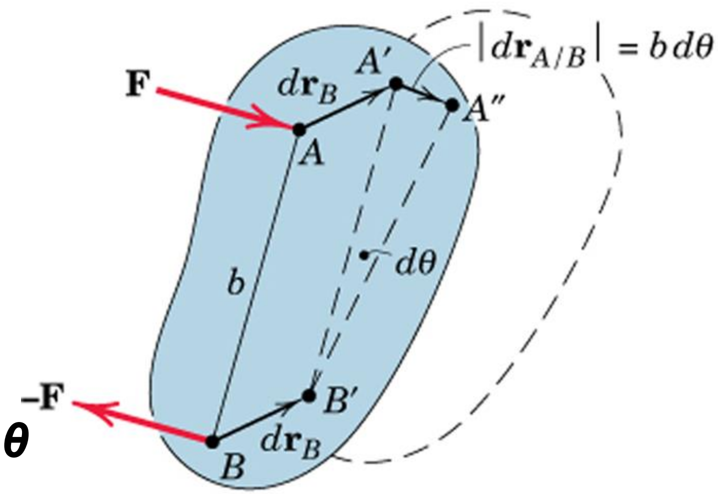
$U = \mathbf{F}.d\mathbf{r}_{A/B} = Fbd\theta$

Since $M = Fb$

→ $$dU = M\,d\theta$$

+*M* → *M* has same sense as *θ*

- *M* →*M* has opp sense as *θ*



Total word done by a couple during a finite rotation in its plane:

$$U = \int M\,d\theta$$

Virtual Work

Dimensions and Units of Work

(Force) x (Distance) → Joule (J) = N.m

→Work done by a force of 1 Newton moving through a distance of 1 m in the direction of the force

→Dimensions of Work of a Force and Moment of a Force are same though they are entirely different physical quantities.

→**Work** is a scalar given by dot product; involves product of a force and distance, both measured along the same line

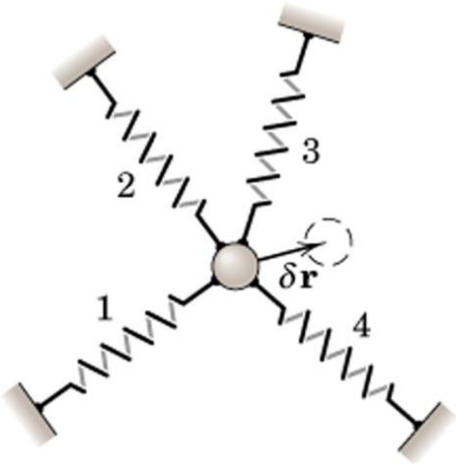
→**Moment** is a vector given by the cross product; involves product of a force and distance measured at right angles to the force

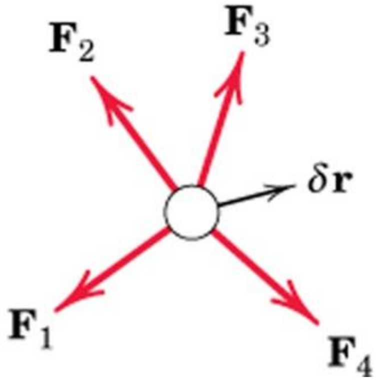
→Units of Work: Joule

→Units of Moment: N.m

Virtual Displacement

Virtual Displacement is not experienced but only **assumed to exist** so that **various possible equilibrium positions** may be **compared** to determine the **correct one**





- *Imagine* the small *virtual displacement* of particle (**δr**) which is acted upon by several forces.

- The corresponding *virtual work*,

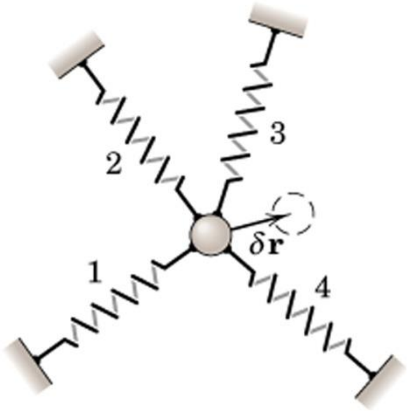
$$\delta U = \vec{F}_1 \cdot \delta\vec{r} + \vec{F}_2 \cdot \delta\vec{r} + \vec{F}_3 \cdot \delta\vec{r} = \left(\vec{F}_1 + \vec{F}_2 + \vec{F}_3\right) \cdot \delta\vec{r}$$

$$= \vec{R} \cdot \delta\vec{r}$$

Virtual Displacement

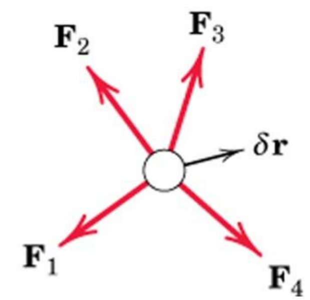
Equilibrium of a Particle

Total virtual work done on the particle due to virtual displacement $\delta\mathbf{r}$:



$$\delta U = \mathbf{F}_1 \cdot \delta\mathbf{r} + \mathbf{F}_2 \cdot \delta\mathbf{r} + \mathbf{F}_3 \cdot \delta\mathbf{r} + \cdots = \Sigma\mathbf{F} \cdot \delta\mathbf{r}$$

Expressing $\sum\mathbf{F}$ in terms of scalar sums and $\delta\mathbf{r}$ in terms of its component virtual displacements in the coordinate directions:



$$\begin{aligned} \delta U &= \Sigma\mathbf{F} \cdot \delta\mathbf{r} = (\mathbf{i}\,\Sigma F_x + \mathbf{j}\,\Sigma F_y + \mathbf{k}\,\Sigma F_z) \cdot (\mathbf{i}\,\delta x + \mathbf{j}\,\delta y + \mathbf{k}\,\delta z) \\ &= \Sigma F_x\,\delta x + \Sigma F_y\,\delta y + \Sigma F_z\,\delta z = 0 \end{aligned}$$

The sum is zero since $\sum\mathbf{F} = 0$, which gives $\sum F_x = 0$, $\sum F_y = 0$, $\sum F_z = 0$

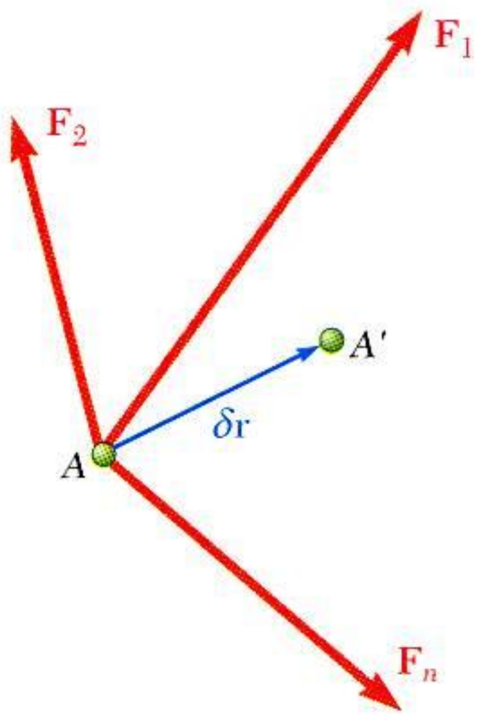
Alternative Statement of the equilibrium: $\delta U = 0$

This condition of **zero virtual work** for **equilibrium** is **both necessary and sufficient** since we can apply it to the **three mutually perpendicular directions** → **3 conditions of equilibrium**

Virtual Work

Principle of Virtual Work:

- If a particle is in equilibrium, the total virtual work of forces acting on the particle is zero for any virtual displacement.

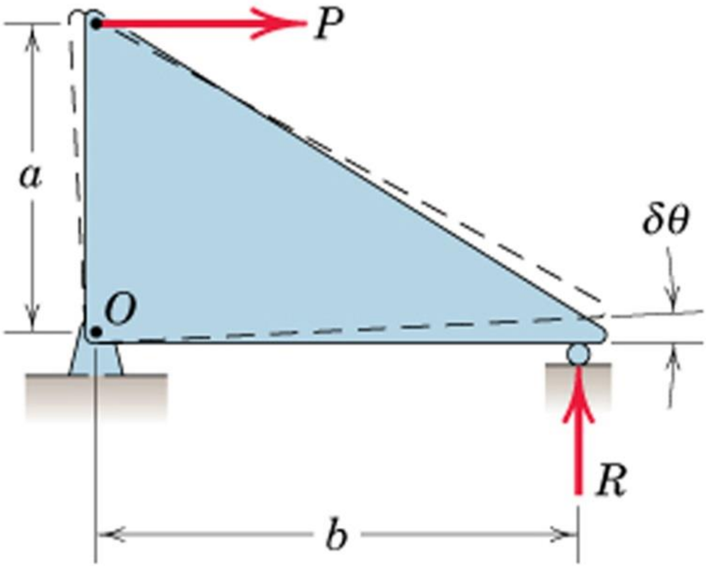


- If a rigid body is in equilibrium
  - total virtual work of external forces acting on the body is zero for any virtual displacement of the body

- If a system of connected rigid bodies remains connected during the virtual displacement
  - the work of the external forces need be considered
    - since work done by internal forces (equal, opposite, and collinear) cancels each other.

Example (1) on Virtual Work Principle

Equilibrium of a Rigid Body



Total virtual work done on the entire rigid body is zero since virtual work done on each Particle of the body in equilibrium is zero.

Weight of the body is negligible.

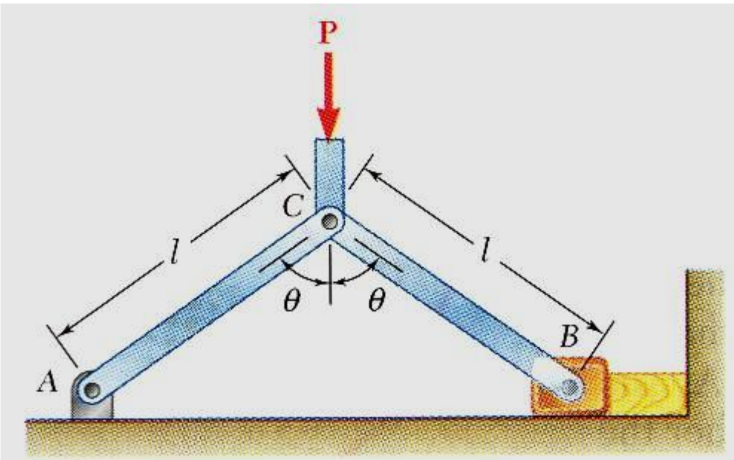
Work done by $P = -Pa\,\delta\theta$
Work done by $R = +Rb\,\delta\theta$

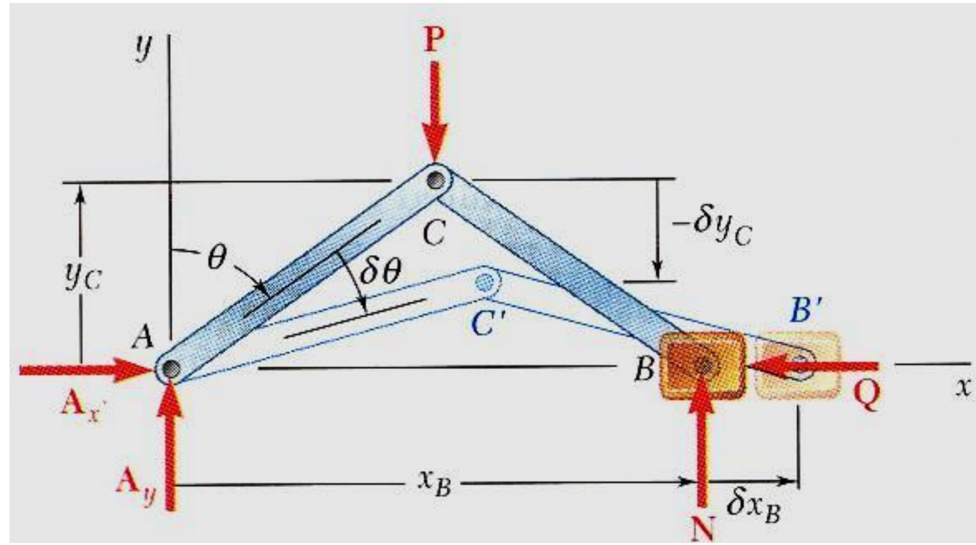
Principle of Virtual Work: $\delta U = 0$:
$-Pa\,\delta\theta + Rb\,\delta\theta = 0$

→ $Pa - Rb = 0$
→ Equation of Moment equilibrium @ O.
→ **Nothing gained** by using the **Principle of Virtual Work** for **a single rigid body**

Example (2) on Virtual Work Principle

Determine the force exerted by the vice on the block when a given force *P* is applied at C. Assume that there is no friction.





- Consider the work done by the external forces for a virtual rotation $\delta\theta$; $\delta\theta$ is a positive increment to θ
- Only the forces *P* and *Q* produce nonzero work.
- x_B increases while y_C decreases

→+ve increment for x_B: $d\,x_B$ → $dU_Q = -\,Qd\,x_B$ (opp. Sense)
→-ve increment for y_C: $-d\,y_C$→$dU_P = +P(-d\,y_C)$ (same Sense)

$$\delta U = 0 = \delta U_Q + \delta U_P = -Q\,\delta x_B - P\,\delta y_C$$

Example (2) on Virtual Work Principle

$$\delta U = 0 = \delta U_Q + \delta U_P = -Q\,\delta x_B - P\,\delta y_C$$

- Expressing x_B and y_C in terms of θ and differentiating w.r.t. θ

$$x_B = 2l\sin\theta \qquad y_C = l\cos\theta$$

$$\delta x_B = 2l\cos\theta\,\delta\theta \qquad \delta y_C = -l\sin\theta\,\delta\theta$$

By using the method of virtual work, all unknown reactions were eliminated. $\sum M_A$ would eliminate only two reactions.

$$0 = -2Ql\cos\theta\,\delta\theta + Pl\sin\theta\,\delta\theta$$

$$Q = \tfrac{1}{2}P\tan\theta$$

- If the virtual displacement is consistent with the constraints imposed by supports and connections, only the work of loads, applied forces, and friction forces need be considered.

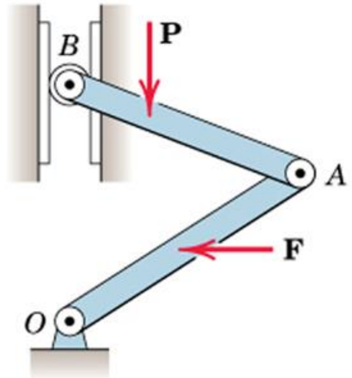
Virtual Work

Principle of Virtual Work

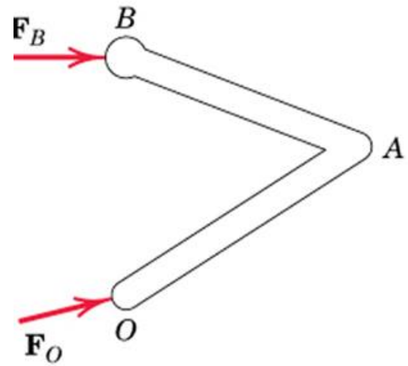
Virtual Work done by external active forces on an ideal mechanical system in equilibrium is zero for any and all virtual displacements consistent with the constraints

$$\delta U = 0$$

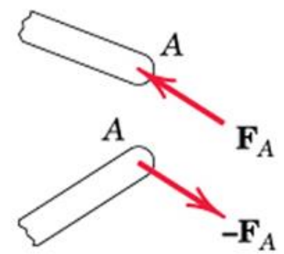
Three types of forces act on interconnected systems made of rigid members



Active Forces: Work Done
Active Force Diagram



Reactive Forces
No Work Done



Internal Forces
No Work Done

Virtual Work

Major Advantages of the Virtual Work Method

- It is not necessary to dismember the systems in order to establish relations between the active forces.
- Relations between active forces can be determined directly without reference to the reactive forces.

→ The method is particularly useful in determining the position of equilibrium of a system under known loads (This is in contrast to determining the forces acting on a body whose equilibrium position is known – studied earlier).

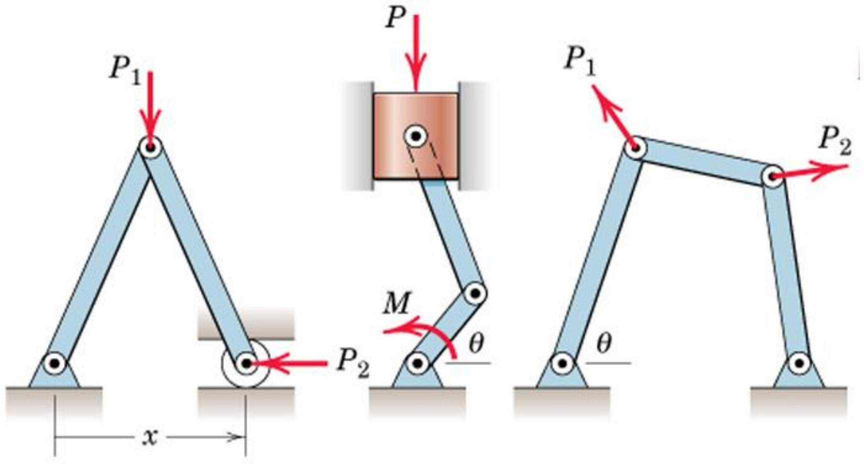
→ The method requires that internal frictional forces do negligible work during any virtual displacement.

→ If internal friction is appreciable, work done by internal frictional forces must be included in the analysis.

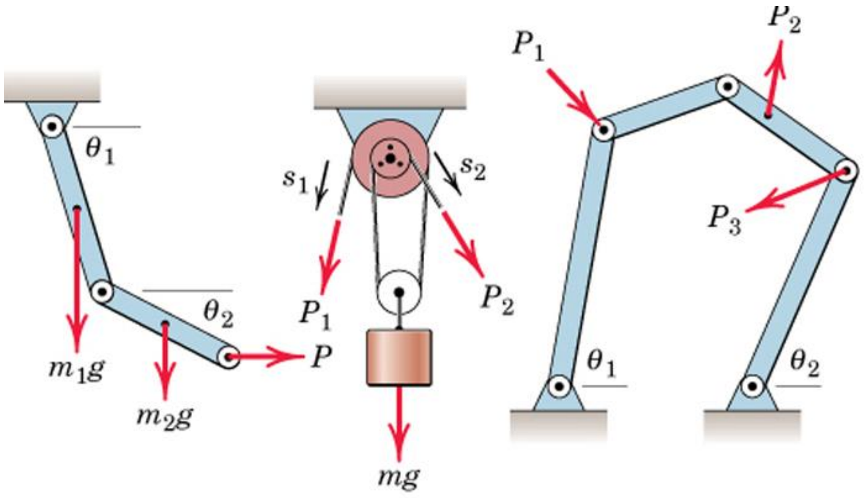
Virtual Work

Degrees of Freedom (DOF)

- Number of independent coordinates needed to specify completely the configuration of system



(a) Examples of one-degree-of-freedom systems



(b) Examples of two-degree-of-freedom systems

Only one coordinate (displacement or rotation) is needed to establish position of every part of the system

Two independent coordinates are needed to establish position of every part of the system

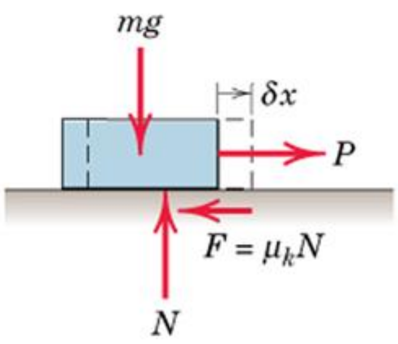
$\delta U = 0$ can be **applied to each DOF at a time keeping other DOF constant.**

ME101 → only SDOF systems

Virtual Work

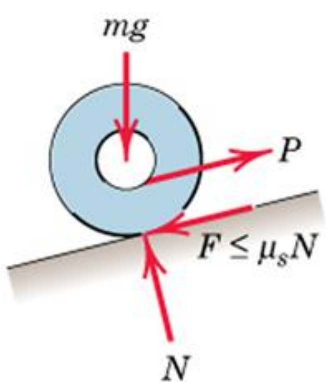
Systems with Friction

- So far, the Principle of virtual work was discussed for “ideal” systems.
- If significant friction is present in the system (“Real” systems), work done by the external active forces (input work) will be opposed by the work done by the friction forces.



During a virtual displacement δx:

Work done by the kinetic friction force is: $-\mu_k N \delta x$



During rolling of a wheel:
the static friction force does no work if the wheel does not slip as it rolls.

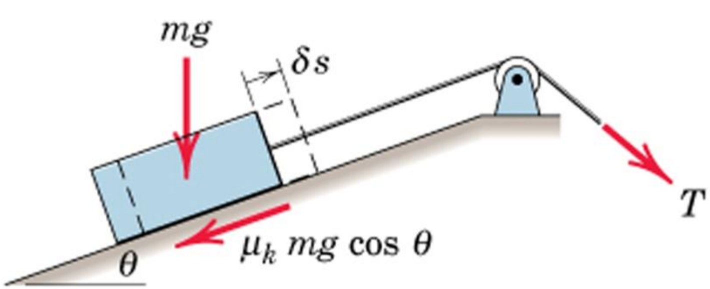
Virtual Work

Mechanical Efficiency (*e*)

- Output work of a machine is always less than the input work because of energy loss due to friction.

$$e = \frac{\text{Output Work}}{\text{Input Work}}$$

For simple machines with SDOF & which operates in uniform manner, mechanical efficiency may be determined using the method of Virtual Work



For the virtual displacement δs: Output Work is that necessary to elevate the block = $mg\, \delta s \sin\theta$

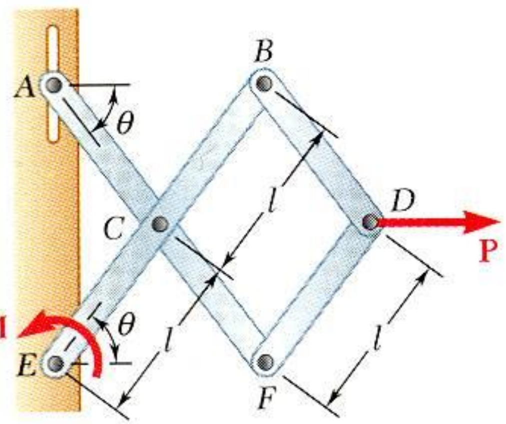
Input Work: $T\, \delta s = mg \sin\theta\, \delta s + \mu_k\, mg \cos\theta\, \delta s$

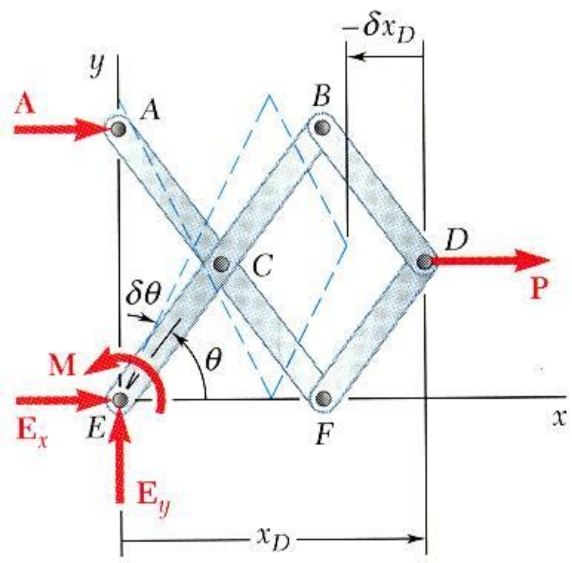
The efficiency of the inclined plane is:

$$e = \frac{mg\,\delta s \sin\theta}{mg(\sin\theta + \mu_k \cos\theta)\delta s} = \frac{1}{1 + \mu_k \cot\theta}$$

As friction decreases, Efficiency approaches unity

Example (3) on Virtual Work





Determine the magnitude of the couple *M* required to maintain the equilibrium of the mechanism.

SOLUTION: Apply a positive virtual increment to θ at E

- Apply the principle of virtual work

$$\delta U = 0 = \delta U_M + \delta U_P$$

$$0 = M\delta\theta + P\delta x_D$$

$\delta\theta$ along positive θ → θ increases with $\delta\theta$ → + $\delta\theta$

M and $\delta\theta$ have same sense → + $M\delta\theta$

For positive $\delta\theta$, x_D will decrease by δx_D → - δx_D

P and - δx_D have opposite sense → - P(- δx_D) → + $P\delta x_D$

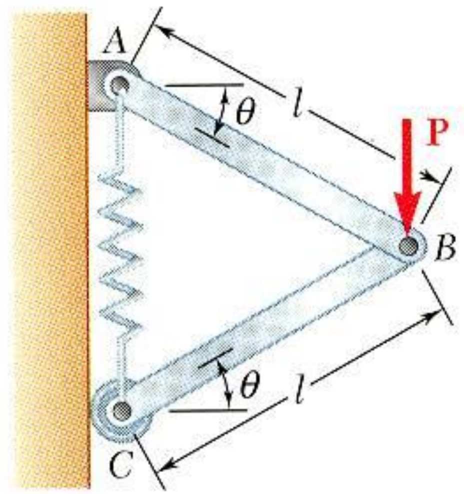
$$x_D = 3l\cos\theta$$

$$\delta x_D = -3l\sin\theta\delta\theta$$

$$0 = M\delta\theta + P(-3l\sin\theta\delta\theta)$$

$$M = 3Pl\sin\theta$$

Example (4) on Virtual Work



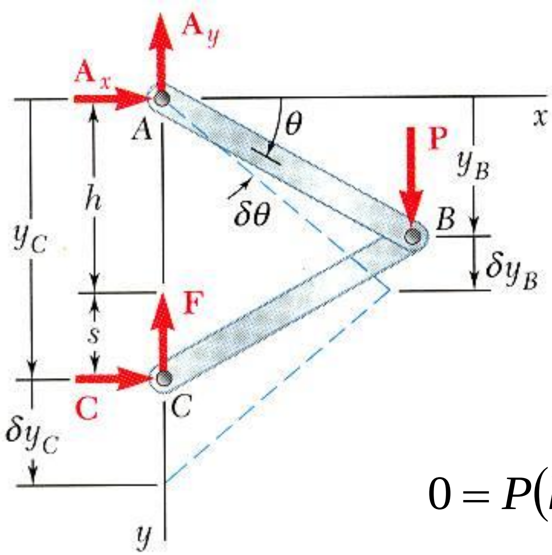
Determine the expressions for θ and for the tension in the spring which correspond to the equilibrium position of the spring. The unstretched length of the spring is *h* and the constant of the spring is *k*. Neglect the weight of the mechanism.

SOLUTION: Apply a positive virtual increment to θ at A

- Apply the principle of virtual work

$$\delta U = \delta U_B + \delta U_F = 0$$

$$0 = P\delta y_B - F\delta y_C$$



y_B increases with δy_B → + δy_B
P and δy_B have same sense → + $P\ \delta y_B$
For positive $\delta\theta$, y_C will increase by δy_C → + δy_C
F and δy_C have opposite sense → -$F(\delta y_C)$ → - $F\delta y_C$

$$y_B = l\sin\theta \qquad y_C = 2l\sin\theta \qquad F = ks$$

$$\delta y_B = l\cos\theta\delta\theta \qquad \delta y_C = 2l\cos\theta\delta\theta \qquad = k(y_C - h)$$

$$= k(2l\sin\theta - h)$$

$$0 = P(l\cos\theta\delta\theta) - k(2l\sin\theta - h)(2l\cos\theta\delta\theta)$$

$$\sin\theta = \frac{P + 2kh}{4kl}$$

$$F = \tfrac{1}{2}P$$