Area Moments of Inertia by Integration

x



 Second moments or moments of inertia of an area with respect to the x and y axes,

$$I_x = \int y^2 dA \qquad I_y = \int x^2 dA$$

• Evaluation of the integrals is simplified by choosing *dA* to be a thin strip parallel to one of the coordinate axes



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Products of Inertia: for problems involving unsymmetrical cross-sections and in calculation of MI about rotated axes. It may be +ve, -ve, or zero



• Product of Inertia of area A w.r.t. x-y axes: $I_{xy} = \int xy \, dA$

x and y are the coordinates of the element of area dA = xy

• When the *x* axis, the *y* axis, or both are an axis of symmetry, the product of inertia is zero.



• Parallel axis theorem for products of inertia:



 $I_{xy} = \bar{I}_{xy} + \bar{x}\bar{y}A$

x'

x

dA

x

0

dA'

Rotation of Axes

Product of inertia is useful in calculating MI @ inclined axes.

 \rightarrow Determination of axes about which the MI is a maximum and a minimum



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Adding first two eqns: $I_{x'} + I_{y'} = I_x + I_y = I_z \rightarrow$ The Polar MI @ O

Angle which makes $I_{x'}$ and $I_{y'}$ either max or min can be found by setting the derivative of either $I_{x'}$ or $I_{y'}$ w.r.t. θ equal to zero:

$$\frac{dI_{x'}}{d\theta} = \left(I_y - I_x\right)\sin 2\theta - 2I_{xy}\cos 2\theta = 0$$

Denoting this critical angle by α

$$\tan 2\alpha = \frac{2I_{xy}}{I_y - I_x}$$

- → two values of 2α which differ by π since $\tan 2\alpha = \tan(2\alpha + \pi)$
- ightarrow two solutions for α will differ by $\pi/2$
- → one value of α will define the axis of maximum MI and the other defines the axis of minimum MI
- \rightarrow These two rectangular axes are called the principal axes of inertia



$$I_{x'} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2} \cos 2\theta - I_{xy} \sin 2\theta$$
$$I_{y'} = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2} \cos 2\theta + I_{xy} \sin 2\theta$$
$$I_{x'y'} = \frac{I_x - I_y}{2} \sin 2\theta + I_{xy} \cos 2\theta$$

$$\tan 2\alpha = \frac{2I_{xy}}{I_y - I_x} \Longrightarrow \sin 2\alpha = \cos 2\alpha \frac{2I_{xy}}{I_y - I_x}$$

Substituting in the third eqn for critical value of 2θ : $I_{x'y'} = 0$

→ Product of Inertia $I_{x'y'}$ is zero for the Principal Axes of inertia

Substituting $\sin 2\alpha$ and $\cos 2\alpha$ in first two eqns for **Principal Moments of Inertia**:

$$I_{\max} = \frac{I_x + I_y}{2} + \frac{1}{2}\sqrt{(I_x - I_y)^2 + 4I_{xy}^2}$$
$$I_{\min} = \frac{I_x + I_y}{2} - \frac{1}{2}\sqrt{(I_x - I_y)^2 + 4I_{xy}^2}$$
$$I_{xy@\alpha} = 0$$

Mohr's Circle of Inertia :: Graphical representation of the MI equations

- For given values of I_x , I_y , & I_{xy} , corresponding values of $I_{x'}$, $I_{y'}$, & $I_{x'y'}$ may be determined from the diagram for any desired angle θ .



$$I_{ave} = \frac{I_x + I_y}{2}$$
 $R = \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2}$

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 $I_{\text{max, min}} = I_{ave} \pm R$

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Mohr's Circle of Inertia: Construction $\tan 2\alpha = \frac{2I_{xy}}{I_y - I_x}$



$$I_{x'} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2} \cos 2\theta - I_{xy} \sin 2\theta$$
$$I_{y'} = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2} \cos 2\theta + I_{xy} \sin 2\theta$$
$$I_{x'y'} = \frac{I_x - I_y}{2} \sin 2\theta + I_{xy} \cos 2\theta$$



Choose horz axis \rightarrow MI Choose vert axis \rightarrow PI Point A – known $\{I_x, I_{xv}\}$ Point B – known $\{I_{v}, -I_{xv}\}$ Circle with dia AB Angle α for Area \rightarrow Angle 2 α to horz (same sense) $\rightarrow I_{max}, I_{min}$ Angle *x* to $x' = \theta$ \rightarrow Angle OA to OC = 2 θ \rightarrow Same sense Point C \rightarrow $I_{x'}$, $I_{x'v'}$ Point D $\rightarrow I_{v'}$

Example: Product of Inertia



SOLUTION:

- Determine the product of inertia using direct integration with the parallel axis theorem on vertical differential area strips
- Apply the parallel axis theorem to evaluate the product of inertia with respect to the centroidal axes.

Determine the product of inertia of the right triangle (*a*) with respect to the *x* and *y* axes and (*b*) with respect to centroidal axes parallel to the *x* and *y* axes.

Examples



SOLUTION:

• Determine the product of inertia using direct integration with the parallel axis theorem on vertical differential area strips

$$y = h \left(1 - \frac{x}{b} \right) \quad dA = y \, dx = h \left(1 - \frac{x}{b} \right) dx$$
$$\overline{x}_{el} = x \qquad \overline{y}_{el} = \frac{1}{2} \, y = \frac{1}{2} h \left(1 - \frac{x}{b} \right)$$

Integrating dI_x from x = 0 to x = b,

$$I_{xy} = \int dI_{xy} = \int \bar{x}_{el} \, \bar{y}_{el} \, dA = \int_{0}^{b} x \left(\frac{1}{2}\right) h^{2} \left(1 - \frac{x}{b}\right)^{2} \, dx$$
$$= h^{2} \int_{0}^{b} \left(\frac{x}{2} - \frac{x^{2}}{b} + \frac{x^{3}}{2b^{2}}\right) dx = h^{2} \left[\frac{x^{2}}{4} - \frac{x^{3}}{3b} + \frac{x^{4}}{8b^{2}}\right]_{0}^{b}$$

 $I_{xy} = \frac{1}{24}b^2h^2$

Examples



SOLUTION

• Apply the parallel axis theorem to evaluate the product of inertia with respect to the centroidal axes.

$$\overline{x} = \frac{1}{3}b \qquad \overline{y} = \frac{1}{3}h$$

With the results from part *a*,

$$I_{xy} = \bar{I}_{x''y''} + \bar{x}\bar{y}A$$
$$\bar{I}_{x''y''} = \frac{1}{24}b^2h^2 - \left(\frac{1}{3}b\right)\left(\frac{1}{3}h\right)\left(\frac{1}{2}bh\right)$$

$$\bar{I}_{x''y''} = -\frac{1}{72}b^2h^2$$



The moments and product of inertia with respect to the x and y axes are $I_x =$ 7.24x106 mm⁴, $I_y = 2.61x106$ mm⁴, and $I_{xy} = -2.54x10^6$ mm⁴.

Using Mohr's circle, determine (a) the principal axes about O, (b) the values of the principal moments about O, and (c) the values of the moments and product of inertia about the x' and y' axes

SOLUTION:

- Plot the points (I_x, I_{xy}) and $(I_y, -I_{xy})$. Construct Mohr's circle based on the circle diameter between the points.
- Based on the circle, determine the orientation of the principal axes and the principal moments of inertia.
- Based on the circle, evaluate the moments and product of inertia with respect to the *x*'*y*' axes.

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Example: Mohr's Circle of Inertia



 $I_x = 7.24 \times 10^6 \text{ mm}^4$ $I_y = 2.61 \times 10^6 \text{ mm}^4$ $I_{xy} = -2.54 \times 10^6 \text{ mm}^4$ SOLUTION:

• Plot the points (I_x, I_{xy}) and $(I_y, -I_{xy})$. Construct Mohr's circle based on the circle diameter between the points.

$$OC = I_{ave} = \frac{1}{2} (I_x + I_y) = 4.925 \times 10^6 \,\mathrm{mm}^4$$
$$CD = \frac{1}{2} (I_x - I_y) = 2.315 \times 10^6 \,\mathrm{mm}^4$$
$$R = \sqrt{(CD)^2 + (DX)^2} = 3.437 \times 10^6 \,\mathrm{mm}^4$$

• Based on the circle, determine the orientation of the principal axes and the principal moments of inertia.

$$\tan 2\theta_m = \frac{DX}{CD} = 1.097 \quad 2\theta_m = 47.6^\circ \qquad \theta_m = 23.8^\circ$$

Example: Mohr's Circle of Inertia





• Based on the circle, evaluate the moments and product of inertia with respect to the *x*'*y*' axes.

 $OC = I_{ave} = 4.925 \times 10^6 \,\mathrm{mm}^4$

 $R = 3.437 \times 10^6 \text{ mm}^4$

The points X' and Y' corresponding to the x' and y' axes are obtained by rotating CX and CY counterclockwise through an angle $\theta = 2(60^\circ) = 120^\circ$. The angle that CX' forms with the horz is $\phi = 120^\circ - 47.6^\circ = 72.4^\circ$.

$$I_{x'} = OF = OC + CX' \cos \varphi = I_{ave} + R \cos 72.4^{\circ}$$

$$I_{x'} = 5.96 \times 10^{6} \text{ mm}^{4}$$

$$I_{y'} = OG = OC - CY' \cos \varphi = I_{ave} - R \cos 72.4^{\circ}$$

$$I_{y'} = 3.89 \times 10^{6} \text{ mm}^{4}$$

$$I_{x'y'} = FX' = CY' \sin \varphi = R \sin 72.4^{\circ}$$

$$I_{x'y'} = 3.28 \times 10^{6} \text{ mm}^{4}$$

^ x'v

- Application in rigid body dynamics
 - Measure of distribution of mass of a rigid body w.r.t. the axis (constant property for that axis)



$$I=\int r^2 dm$$

r = perpendicular distance of the mass element dm from the axis O-O

 $r^2 \Delta m$:: measure of the inertia of the system

• About individual coordinate axes



$$\begin{split} I_x &= \int \left(y^2 + z^2\right) \, dm \\ I_y &= \int \left(z^2 + x^2\right) \, dm \\ I_z &= \int \left(x^2 + y^2\right) \, dm \end{split}$$

• Parallel Axis Theorem

$$y' \qquad x = x' + \overline{x} \qquad y = y' + \overline{y} \qquad z = z' + I_x = \overline{I}_{x'} + m(\overline{y}^2 + \overline{z}^2)$$

$$I_x = \overline{I}_{x'} + m(\overline{y}^2 + \overline{z}^2)$$

$$I_y = \overline{I}_{y'} + m(\overline{z}^2 + \overline{x}^2)$$

$$I_z = \overline{I}_{z'} + m(\overline{x}^2 + \overline{y}^2)$$

z

 \overline{z}

Moments of Inertia of Thin Plates

For a thin plate of uniform thickness *t* and homogeneous material of density *ρ*, the mass moment of inertia with respect to axis *AA*' contained in the plate is

$$I_{AA'} = \int r^2 dm = \rho t \int r^2 dA$$
$$= \rho t I_{AA',area}$$

- B OdA OB'
- Similarly, for perpendicular axis *BB*' which is also contained in the plate,
 - $I_{BB'} = \rho t \, I_{BB',area}$
- For the axis *CC*' which is perpendicular to the plate, $I_{CC'} = \rho t J_{C,area} = \rho t (I_{AA',area} + I_{BB',area})$ $= I_{AA'} + I_{BB'}$

A'

C

Moments of Inertia of Thin Plates



• For the principal centroidal axes on a rectangular plate,

$$I_{AA'} = \rho t I_{AA',area} = \rho t \left(\frac{1}{12}a^3b\right) = \frac{1}{12}ma^2$$
$$I_{BB'} = \rho t I_{BB',area} = \rho t \left(\frac{1}{12}ab^3\right) = \frac{1}{12}mb^2$$
$$I_{CC'} = I_{AA',mass} + I_{BB',mass} = \frac{1}{12}m(a^2 + b^2)$$



• For centroidal axes on a circular plate,

$$I_{AA'} = I_{BB'} = \rho t I_{AA',area} = \rho t \left(\frac{1}{4}\pi r^4\right) = \frac{1}{4}mr^2$$

Moments of Inertia of a 3D Body by Integration



• Moment of inertia of a homogeneous body is obtained from double or triple integrations of the form

$$I = \rho \int r^2 dV$$

- For bodies with two planes of symmetry, the moment of inertia may be obtained from a single integration by choosing thin slabs perpendicular to the planes of symmetry for *dm*.
- The moment of inertia with respect to a particular axis for a composite body may be obtained by adding the moments of inertia with respect to the same axis of the components.



