Area Moments of Inertia by Integration

- **Second moments or moments of inertia** of an area with respect to the \( x \) and \( y \) axes,

\[
I_x = \int y^2 \, dA \quad I_y = \int x^2 \, dA
\]

- Evaluation of the integrals is simplified by choosing \( dA \) to be a thin strip parallel to one of the coordinate axes.
Area Moments of Inertia

**Products of Inertia:** for problems involving unsymmetrical cross-sections and in calculation of MI about rotated axes. It may be +ve, -ve, or zero

- **Product of Inertia of area** $A$ w.r.t. $x$-$y$ axes:
  \[ I_{xy} = \int xy\,dA \]
  $x$ and $y$ are the coordinates of the element of area $dA=xy$

- When the $x$ axis, the $y$ axis, or both are an axis of symmetry, the product of inertia is zero.

- **Parallel axis theorem for products of inertia:**
  \[ I_{xy} = \bar{I}_{xy} + \bar{x}\bar{y}A \]

**Quadrants**
Area Moments of Inertia

Rotation of Axes

Product of inertia is useful in calculating MI @ inclined axes.

→ Determination of axes about which the MI is a maximum and a minimum

\[
I_x' = \int y'^2 dA = \int (y \cos \theta - x \sin \theta)^2 dA
\]

\[
I_y' = \int x'^2 dA = \int (x \cos \theta + y \sin \theta)^2 dA
\]

\[
I_{x'y'} = \int x'y' dA = \int (x \cos \theta + y \sin \theta)(y \cos \theta - x \sin \theta) dA
\]

\[
\sin^2 \theta = \frac{1 - \cos 2\theta}{2} \quad \cos^2 \theta = \frac{1 + \cos 2\theta}{2}
\]

\[
\sin \theta \cos \theta = \frac{1}{2} \sin 2\theta \quad \cos^2 \theta - \sin^2 \theta = \cos 2\theta
\]

Moments and product of inertia w.r.t. new axes \(x'\) and \(y'\)?

Note: \[ x' = x \cos \theta + y \sin \theta \]

\[ y' = y \cos \theta - x \sin \theta \]

\[
I_x = \int y^2 dA \quad I_y = \int x^2 dA
\]

\[
I_{xy} = \int xy dA
\]
Area Moments of Inertia

Adding first two eqns: 
\[ I_x' + I_y' = I_x + I_y = I_z \] → The Polar MI @ O

Angle which makes \( I_x' \) and \( I_y' \) either max or min can be found by setting the derivative of either \( I_x' \) or \( I_y' \) w.r.t. \( \theta \) equal to zero:

\[ \frac{dI_{x'}}{d\theta} = (I_y - I_x)\sin 2\theta - 2I_{xy}\cos 2\theta = 0 \]

Denoting this critical angle by \( \alpha \)

\[ \tan 2\alpha = \frac{2I_{xy}}{I_y - I_x} \]

→ two values of \( 2\alpha \) which differ by \( \pi \) since \( \tan 2\alpha = \tan(2\alpha + \pi) \)
→ two solutions for \( \alpha \) will differ by \( \pi/2 \)
→ one value of \( \alpha \) will define the axis of maximum MI and the other defines the axis of minimum MI
→ These two rectangular axes are called the principal axes of inertia
Area Moments of Inertia

Rotation of Axes

\[ I_{x'} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2} \cos 2\theta - I_{xy} \sin 2\theta \]

\[ I_{y'} = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2} \cos 2\theta + I_{xy} \sin 2\theta \]

\[ I_{x'y'} = \frac{I_x - I_y}{2} \sin 2\theta + I_{xy} \cos 2\theta \]

\[ \tan 2\alpha = \frac{2I_{xy}}{I_y - I_x} \Rightarrow \sin 2\alpha = \cos 2\alpha \frac{2I_{xy}}{I_y - I_x} \]

Substituting in the third eqn for critical value of 2\theta: \( I_{x'y'} = 0 \)

\[ \Rightarrow \text{Product of Inertia } I_{x'y'} \text{ is zero for the Principal Axes of inertia} \]

Substituting \( \sin 2\alpha \) and \( \cos 2\alpha \) in first two eqns for Principal Moments of Inertia:

\[ I_{max} = \frac{I_x + I_y}{2} + \frac{1}{2} \sqrt{(I_x - I_y)^2 + 4I_{xy}^2} \]

\[ I_{min} = \frac{I_x + I_y}{2} - \frac{1}{2} \sqrt{(I_x - I_y)^2 + 4I_{xy}^2} \]

\[ I_{xy@\alpha} = 0 \]
Area Moments of Inertia

Mohr’s Circle of Inertia :: Graphical representation of the MI equations

- For given values of $I_x$, $I_y$, & $I_{xy}$, corresponding values of $I_{x'}$, $I_{y'}$, & $I_{x'y'}$ may be determined from the diagram for any desired angle $\theta$.

\[
I_x' = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2} \cos 2\theta - I_{xy} \sin 2\theta
\]

\[
I_y' = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2} \cos 2\theta + I_{xy} \sin 2\theta
\]

\[
I_{x'y'} = \frac{I_x - I_y}{2} \sin 2\theta + I_{xy} \cos 2\theta
\]

\[
I_{max} = \frac{I_x + I_y}{2} + \frac{1}{2} \sqrt{(I_x - I_y)^2 + 4I_{xy}^2}
\]

\[
I_{min} = \frac{I_x + I_y}{2} - \frac{1}{2} \sqrt{(I_x - I_y)^2 + 4I_{xy}^2}
\]

\[
I_{xy@\alpha} = 0
\]

\[
(I_x' - I_{ave})^2 + I_{x'y'}^2 = R^2
\]

\[
I_{ave} = \frac{I_x + I_y}{2}
\]

\[
R = \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2}
\]

- At the points $A$ and $B$, $I_{x'y'} = 0$ and $I_x'$ takes the maximum and minimum values

\[
I_{max, min} = I_{ave} \pm R
\]
Area Moments of Inertia

Mohr’s Circle of Inertia: Construction

\[ \tan 2\alpha = \frac{2I_{xy}}{I_y - I_x} \]

Choose horz axis \( \rightarrow \) MI
Choose vert axis \( \rightarrow \) PI
Point A – known \( \{I_x, I_{xy}\} \)
Point B – known \( \{I_y, -I_{xy}\} \)
Circle with dia AB
Angle \( \alpha \) for Area
\( \rightarrow \) Angle \( 2\alpha \) to horz (same sense) \( \rightarrow \) \( I_{\text{max}}, I_{\text{min}} \)
Angle \( x \) to \( x' = \theta \)
\( \rightarrow \) Angle OA to OC = \( 2\theta \)
\( \rightarrow \) Same sense
Point C \( \rightarrow I_{x'}, I_{x'y'} \)
Point D \( \rightarrow I_{y'} \)

\[
I_x' = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2} \cos 2\theta - I_{xy} \sin 2\theta
\]

\[
I_y' = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2} \cos 2\theta + I_{xy} \sin 2\theta
\]

\[
I_{xy}' = \frac{I_y - I_x}{2} \sin 2\theta + I_{xy} \cos 2\theta
\]

\[
I_{\text{max}} = \frac{I_x + I_y}{2} + \frac{1}{2} \sqrt{(I_x - I_y)^2 + 4I_{xy}^2}
\]

\[
I_{\text{min}} = \frac{I_x + I_y}{2} - \frac{1}{2} \sqrt{(I_x - I_y)^2 + 4I_{xy}^2}
\]

\[
I_{xy@\alpha} = 0
\]
Example: Product of Inertia

Determine the product of inertia of the right triangle $(a)$ with respect to the $x$ and $y$ axes and $(b)$ with respect to centroidal axes parallel to the $x$ and $y$ axes.

SOLUTION:

- Determine the product of inertia using direct integration with the parallel axis theorem on vertical differential area strips.
- Apply the parallel axis theorem to evaluate the product of inertia with respect to the centroidal axes.
Area Moments of Inertia

Examples

SOLUTION:

- Determine the product of inertia using direct integration with the parallel axis theorem on vertical differential area strips

\[
y = h \left(1 - \frac{x}{b}\right) \quad dA = y \, dx = h \left(1 - \frac{x}{b}\right) \, dx
\]

\[
\bar{x}_{el} = x \quad \bar{y}_{el} = \frac{1}{2} y = \frac{1}{2} h \left(1 - \frac{x}{b}\right)
\]

Integrating \(dI_x\) from \(x = 0\) to \(x = b\),

\[
I_{xy} = \int dI_{xy} = \int x \, dx\bar{y}_{el} \, dA = \int x \left(\frac{1}{2}\right) h^2 \left(1 - \frac{x}{b}\right)^2 \, dx
\]

\[
= h^2 \left[ \frac{x^2}{2} - \frac{x^2}{b} + \frac{x^3}{2b^2} \right]_0^b = h \left[ \frac{x^2}{4} - \frac{x^3}{3b} + \frac{x^4}{8b^2} \right]_0^b
\]

\[
I_{xy} = \frac{1}{24} b^2 h^2
\]
Area Moments of Inertia

Examples

SOLUTION

- Apply the parallel axis theorem to evaluate the product of inertia with respect to the centroidal axes.

\[ \bar{x} = \frac{1}{3} b \quad \bar{y} = \frac{1}{3} h \]

With the results from part a,

\[ I_{xy} = \bar{I}_{x'y'} + \bar{x}\bar{y}A \]

\[ \bar{I}_{x'y'} = \frac{1}{24} b^2 h^2 - \left(\frac{1}{3} b\right)\left(\frac{1}{3} h\right)\left(\frac{1}{2} bh\right) \]

\[ \bar{I}_{x'y'} = -\frac{1}{72} b^2 h^2 \]
The moments and product of inertia with respect to the \( x \) and \( y \) axes are \( I_x = 7.24 \times 10^6 \, \text{mm}^4 \), \( I_y = 2.61 \times 10^6 \, \text{mm}^4 \), and \( I_{xy} = -2.54 \times 10^6 \, \text{mm}^4 \).

Using Mohr’s circle, determine (a) the principal axes about \( O \), (b) the values of the principal moments about \( O \), and (c) the values of the moments and product of inertia about the \( x' \) and \( y' \) axes.

SOLUTION:

- Plot the points \((I_x, I_{xy})\) and \((I_y, -I_{xy})\). Construct Mohr’s circle based on the circle diameter between the points.
- Based on the circle, determine the orientation of the principal axes and the principal moments of inertia.
- Based on the circle, evaluate the moments and product of inertia with respect to the \( x' \) and \( y' \) axes.
Area Moments of Inertia

Example: Mohr’s Circle of Inertia

**SOLUTION:**
- Plot the points \((I_x, I_{xy})\) and \((I_y, -I_{xy})\). Construct Mohr’s circle based on the circle diameter between the points.

\[
OC = I_{ave} = \frac{1}{2}(I_x + I_y) = 4.925 \times 10^6 \text{ mm}^4
\]
\[
CD = \frac{1}{2}(I_x - I_y) = 2.315 \times 10^6 \text{ mm}^4
\]
\[
R = \sqrt{(CD)^2 + (DX)^2} = 3.437 \times 10^6 \text{ mm}^4
\]

- Based on the circle, determine the orientation of the principal axes and the principal moments of inertia.

\[
\tan 2\theta_m = \frac{DX}{CD} = 1.097 \quad 2\theta_m = 47.6^\circ \quad \theta_m = 23.8^\circ
\]

\(I_x = 7.24 \times 10^6 \text{ mm}^4\)
\(I_y = 2.61 \times 10^6 \text{ mm}^4\)
\(I_{xy} = -2.54 \times 10^6 \text{ mm}^4\)
Area Moments of Inertia

Example: Mohr’s Circle of Inertia

- Based on the circle, evaluate the moments and product of inertia with respect to the $x'y'$ axes.

The points $X'$ and $Y'$ corresponding to the $x'$ and $y'$ axes are obtained by rotating $CX$ and $CY$ counterclockwise through an angle $\theta = 2(60^\circ) = 120^\circ$. The angle that $CX'$ forms with the horz is $\phi = 120^\circ - 47.6^\circ = 72.4^\circ$.

\[ I_{x'} = OF = OC + CX'\cos \phi = I_{ave} + R\cos 72.4^\circ \]

\[ I_{x'} = 5.96 \times 10^6 \text{ mm}^4 \]

\[ I_{y'} = OG = OC - CY'\cos \phi = I_{ave} - R\cos 72.4^\circ \]

\[ I_{y'} = 3.89 \times 10^6 \text{ mm}^4 \]

\[ I_{x'y'} = FX' = CY'\sin \phi = R\sin 72.4^\circ \]

\[ I_{x'y'} = 3.28 \times 10^6 \text{ mm}^4 \]
Mass Moment of Inertia

- Application in rigid body dynamics
  - Measure of distribution of mass of a rigid body w.r.t. the axis (constant property for that axis)

\[ I = \int r^2 \, dm \]

\[ r = \text{perpendicular distance of the mass element } dm \text{ from the axis } O-O \]

\[ r^2 \Delta m \text{ :: measure of the inertia of the system} \]
Mass Moment of Inertia

- About individual coordinate axes

\[ I_x = \int (y^2 + z^2) \, dm \]
\[ I_y = \int (z^2 + x^2) \, dm \]
\[ I_z = \int (x^2 + y^2) \, dm \]
Mass Moment of Inertia

- Parallel Axis Theorem

\[
x = x' + \bar{x} \quad y = y' + \bar{y} \quad z = z' + \bar{z}
\]

\[
I_x = I_{x'} + m(\bar{y}^2 + \bar{z}^2)
\]

\[
I_y = I_{y'} + m(\bar{z}^2 + \bar{x}^2)
\]

\[
I_z = I_{z'} + m(\bar{x}^2 + \bar{y}^2)
\]
Mass Moment of Inertia

Moments of Inertia of Thin Plates

• For a thin plate of uniform thickness \( t \) and homogeneous material of density \( \rho \), the mass moment of inertia with respect to axis \( AA' \) contained in the plate is

\[
I_{AA'} = \int r^2 dm = \rho t \int r^2 dA
\]

\[
= \rho t I_{AA', area}
\]

• Similarly, for perpendicular axis \( BB' \) which is also contained in the plate,

\[
I_{BB'} = \rho t I_{BB', area}
\]

• For the axis \( CC' \) which is perpendicular to the plate,

\[
I_{CC'} = \rho t J_{C, area} = \rho t \left( I_{AA', area} + I_{BB', area} \right)
\]

\[
= I_{AA'} + I_{BB'}
\]
Mass Moment of Inertia

Moments of Inertia of Thin Plates

- For the principal centroidal axes on a rectangular plate,
  \[ I_{AA'} = \rho t I_{AA',area} = \rho t \left( \frac{1}{12} a^3 b \right) = \frac{1}{12} ma^2 \]
  \[ I_{BB'} = \rho t I_{BB',area} = \rho t \left( \frac{1}{12} ab^3 \right) = \frac{1}{12} mb^2 \]
  \[ I_{CC'} = I_{AA',mass} + I_{BB',mass} = \frac{1}{12} m \left( a^2 + b^2 \right) \]

- For centroidal axes on a circular plate,
  \[ I_{AA'} = I_{BB'} = \rho t I_{AA',area} = \rho t \left( \frac{1}{4} \pi r^4 \right) = \frac{1}{4} mr^2 \]
Mass Moment of Inertia

Moments of Inertia of a 3D Body by Integration

- Moment of inertia of a homogeneous body is obtained from double or triple integrations of the form

\[ I = \rho \int r^2 dV \]

- For bodies with two planes of symmetry, the moment of inertia may be obtained from a single integration by choosing thin slabs perpendicular to the planes of symmetry for \( dm \).

- The moment of inertia with respect to a particular axis for a composite body may be obtained by adding the moments of inertia with respect to the same axis of the components.

\[
\begin{align*}
  dm &= \rho \pi r^2 dx \\
  dI_x &= \frac{1}{2} r^2 dm \\
  dI_y &= dI_y' + x^2 dm = \left( \frac{1}{4} r^2 + x^2 \right) dm \\
  dI_z &= dI_z' + x^2 dm = \left( \frac{1}{4} r^2 + x^2 \right) dm
\end{align*}
\]
Mass Moment of Inertia

MI of some common geometric shapes

<table>
<thead>
<tr>
<th>Shape</th>
<th>( I_x )</th>
<th>( I_y )</th>
<th>( I_z )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rod ( L )</td>
<td>( \frac{1}{12} mL^2 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Rectangular Plate</td>
<td>( \frac{1}{12} m(b^2 + c^2) )</td>
<td>( \frac{1}{12} mc^2 )</td>
<td>( \frac{1}{12} mb^2 )</td>
</tr>
<tr>
<td>Cylinder ( L )</td>
<td>( \frac{1}{2} ma^2 )</td>
<td></td>
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